

Averaging of one-parameter semigroups and passage to the limit in the space of pseudomeasures

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Abstract

The sequence of one-parameter semigroups arising as the approximation of initial-boundary value problem with singularities is the object of investigation of this paper. The set of limit points of the sequence of approximating semigroups is studied. The set of limit points of the map with values in a linear topological space is presented as the set of mean values of this map by measures on the domain of definition of the map. One to one correspondence between the semigroups generated by any approximating initial-boundary value problems and the pseudomeasures on the space of maps of time semiaxis into the coordinate space is studied. The linear space of pseudomeasures endowed with the structure of Banach space and with the structure of the linear topological space such that the convergence of semigroup sequence is equivalent to the convergence of the sequence of corresponding pseudomeasures. The description of a limit point of the sequence of approximating semigroups is obtained by a measure on the topological vector space of corresponding pseudomeasures. The trajectories the limit one-parameter family of transformations of the space of initial data is described by the mean value of the random pseudomeasure.

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1 Introduction

The averaging of the set of elements of some linear space is the integral of identical map of the linear space onto itself by some nonnegative normalized measure with support on this set. The averaging procedure in the topological linear spaces of pseudomeasures or operator-valued functions is investigated. This procedure is applied to the description of approximation of Cauchy problem with the destruction (see [10, 11, 13, 30]) of solution by the sequence of Cauchy problems in some space of operators presenting the Cauchy problems.

The notion of statistical solution was introduced in the papers by [2], [28], [27] for the investigation of the initial value problems without the uniqueness of solution. For studying of this class of problems the Cauchy problem for the maps of time interval into the space of measures on the space of initial data H is considered instead of the Cauchy problem for the map of time interval into the space of initial data. The another transformation of the Cauchy problem for the map of time interval into the space of initial data H is to define the measure on the Banach space of solutions of Cauchy problem ($C(R_+, H)$ for example).

The initial-boundary value problem without the existence of solution can be transformed into the problem of description of the limit points set for the sequence of solution of approximating initial-boundary value problems. One of the tools of the description of this limit points set is the measure on the space of solution of approximating initial-boundary value problems which presents the distribution of values of this sequence on the space of solution, see [12].

The investigation of such limit measures on the space of solutions had been studied in the paper [19]. In this paper the barycenter of a limit measure is described as the solution of some variational problem on the space of solutions. The procedure of continuation of the resolution map of the problem by the baricenter of limit measure on the space of solutions is similar to the procedure of averaging (homogenization) of differential operators (see [30]).

The procedure of averaging (homogenization) of differential operators is defined in [30] by the passage to the limit for the sequence of solutions of initial-boundary-value problems depending on some small parameters. In this book there is the posing of the new initial-boundary valued problem for some new differential equation such that the limit point of the sequence is the solution of this problem. We show that the limit point of the sequence of solutions can be described as the barycenter of some measure on the space of solutions.

In this paper as in the work [20] the map of the set of contractive semigroups of bounded linear operators of Hilbert space $H = L_2(R^d)$ into the space of pseudomeasures on the space of functions on time interval with values in the coordinate space R^d is defined. In the space of pseudomeasures the structure of Banach space is introduced.

The parametrization of the set of all limit points of the sequence of solutions of regularizing problems by the family of nonnegative normalized measures on the set of regularizing problems

is obtained. The relation between the limit points of the sequence of regularizing semigroups and the limit points of the sequence of corresponding pseudomeasures is studied. It will be showed that the limit point of pseudomeasure sequence is the pseudomeasure but the limit point of the semigroup sequence can't be the semigroup.

The parametrization of the of limit points for the sequence of pseudomeasures is given by the set of normalised nonnegative measures on the elements of the sequence. In particular this description includes the set of all generalised Banach limits of the the sequence (see [21], [7]) since any Banach limit of the sequence can be parametrized by the measure on the sequence elements which is invariant with respect to some group of its transformations.

The set of limit points of of the sequence of semigroups (of pseudomeasures) is endowed with the structure of the space with the measure by the introduction of the measure on the elements of the sequence of approximating semigroups (pseudomeasures). This construction gives the opportunity for continuation of the solution of initial-boundary value problem accross the moment of arising of singularities, bifurcation or blow-up by using of the random variable with values in the set of semigroups of transformation of the space of initial data (or by using of the random variable with values in the set of pseudomeasures on the set of maps of time intervale of the problem into its coordinate space). Thus the arising of the stochastic properties in the dynamical maps of initial-boundary value problem with the effect of destruction of solution or its nonuniqueness is based on the behavior of approximations of considered initial-boundary value problem in the space of such problems.

The structure of the paper is following. Firstly the theorem of presentation of the set of limit points of a map with values in topological vector space by the set of mean values of this map with respect to arbitrary measures on the domain of this map has been proved. The second aim of this article is the relation between the semigroups of transformations of a Hilbert space and pseudomeasures on the space of maps of of time semiaxe into the coordinate space of the initial boundary value problem. The linear space of pseudomeasures is endowed with the topology and with the norm which corresponds to some topologies in the space of operator-valued maps on time semiaxe. The third aim of the article is the random variables with the values in the topological vector spaces of pseudomeasures or operator-valued functions with different topologies. The relation between the measure on the space of pseudomeasures or on the space of operator-valued functions with the Young measures is considered (see [6, 8]).

2 The limit points and the averaging of generalized sequences

. Let E be the topological space and 2^E is the algebra of all subsets of the set E . Let $W(E)$ be the set of nonnegative normalized finite additive measures on the measurable space $(E, 2^E)$. For any point $\varepsilon_0 \in E$ the symbol $W(E, \varepsilon_0)$ notes the set of nonnegative normalized finite additive measures on the measurable space $(E, 2^E)$ such that any measure $\mu \in W(E, \varepsilon_0)$ is concentrated in arbitrary neighborhood of the point ε_0 in the following sence: the equality $\mu(A) = 0$ holds for any set $A \subset E$ such that ε_0 is not limit point of the set A . Let $W_0(E, \varepsilon_0) = \{\mu \in W(E, \varepsilon_0) : \mu(A) \in \{0, 1\} \forall A \in 2^E\}$ be the subset of two-valued measures in the set $W(E, \varepsilon_0)$.

Let S, Z be two linear spaces and $\beta : S \times Z \rightarrow C$ is the bilinear form $\beta : \beta(z, s) = \langle z, s \rangle = z(s)$ which defines the separated duality relation on the spaces S, Z (see [22]): if $z(s) = 0 \forall s \in S$ then the equality $z = \theta_Z$ holds; conversely, if $z(s) = 0 \forall z \in Z$, then $s = \theta_S$. Therefore weak topology $\sigma(Z, S)$ on the space Z and weak topology $\sigma(S, Z)$ on the space S are Hausdorff topology.

Linear functional f on the space S is continuous in $\sigma(S, Z)$ topology if and only if there is the unique element $z_f \in Z$ such that $f(s) = z_f(s)$ for any $s \in S$ (see [22], p. 159, statement 1.2). The reverce statement is true: the linear functional f on the space Z is continuous in $\sigma(Z, S)$ -topology iff there is the unique element $s_f \in S$ such that $f(z) = \langle z, s_f \rangle$ for any $z \in Z$.

Theorem 1. *Let S, Z be some two linear spaces. Suppose that $\beta : S \times Z \rightarrow C$ is the bilinear form $\beta : \beta(z, s) = \langle z, s \rangle = z(s)$ which defines the separated duality relation on the spaces S, Z . Let $G : E \rightarrow Z$ be some map of topological space E into the linear topological space $(Z, \sigma(Z, S))$. If the set $G(E)$ is bounded in the space $(Z, \sigma(Z, S))$ then*

$$\text{Ls}_{\varepsilon \rightarrow \varepsilon_0} G(\varepsilon) = \bigcup_{\mu \in W_0(E, \varepsilon_0)} \int_E G(\varepsilon) d\mu, \quad (1)$$

where $\text{Ls}_{\varepsilon \rightarrow \varepsilon_0} G(\varepsilon)$ is the set of limit points of the map G as $\varepsilon \rightarrow \varepsilon_0$ and integral in right hand side is Pettis integral $\int_E G(\varepsilon) d\mu = g \in Z \Leftrightarrow \varphi(g) = \int_E \varphi(G(\varepsilon)) d\mu \forall \varphi \in S$.

If the limit $\lim_{\varepsilon \rightarrow \varepsilon_0} G(\varepsilon) = g \in Z$ exists then $g = \int_E G(\varepsilon) d\mu$ for any $\mu \in W(E)$.

In addition if the map G is continuous in some deleted neighborhood of the point ε_0 and the set E is arcwise connected then

$$\text{Ls}_{\varepsilon \rightarrow \varepsilon_0} G(\varepsilon) = \bigcup_{\mu \in W(E, \varepsilon_0)} \int_E G(\varepsilon) d\mu.$$

Proof. For any functional $\varphi \in S$ the function $\langle G(\varepsilon), \varphi \rangle, \varepsilon \in E$, is bounded. Then for any measure $\mu \in W(E)$ Pettis integral $I_{\mu, G}(\varphi) = \int_E \varphi(G(\varepsilon)) d\mu$ is defined and the map $I_{\mu, G}$ is

linear with respect to φ . Then the functional $I_{\mu,G}$ is linear functional on the space S . It is obviously continuous in $\sigma(S, Z)$ topology. Hence there is the unique element $g_{\mu,G} \in Z$ such that $\varphi(g_{\mu,G}) = g_{\mu,G}(\varphi)$ for all $\varphi \in S$. Thus for any $\mu \in W(E)$ Pettis integral defines the unique point $\int_E G(\varepsilon) d\mu = g_{\mu,G} \in Z$.

If $\mu \in W(E, \varepsilon_0)$ then system of sets $\mu^{-1}(1) \subset 2^E$ is the ultrafilter F_μ of subsets of the set E . For any $F \in F_\mu$ the point ε_0 is the limit point of the set F in the topology space E . Hence for any functional $\varphi \in S$ integral of the function $\varphi \circ G$ by measure μ is equal to the limit of the function $\varphi \circ G$ along the ultrafilter F_μ since for any $\delta > 0$ the inclusion $(\varphi \circ G)^{-1}(g_{\mu,G}(\varphi) - \delta, g_{\mu,G}(\varphi) + \delta) \in F_\mu$ holds according to the definition of Pettis integral.

The topology $\tau(Z, S)$ is generated by the family of neighborhoods $\{O_{\varepsilon,\varphi}(g), \varepsilon > 0, \varphi \in S, g \in Z\}$ which is the base of topology $\tau(Z, S)$.

According to definition of the point $g_{\mu,G}$ for any $\varepsilon > 0$ and any $\varphi \in S$ the inclusion $G^{-1}[O_{\varepsilon,\varphi}(g_{\mu,G})] \in F_\mu$ holds. Hence the limit of the map G in the topology $\tau(Z, S)$ along the ultrafilter F_μ is equal to $g_{\mu,G}$. Therefore for any measure $\mu \in W_0(E)$ the point $g_{\mu,G}$ is the limit point of the map G as $\varepsilon \rightarrow \varepsilon_0$ i.e. $\bigcup_{\mu \in W_0(E)} \int_E G d\mu \subset \text{Ls}_{\varepsilon \rightarrow \varepsilon_0} G(\varepsilon)$.

Let g be the limit point of the map G as $\varepsilon \rightarrow \varepsilon_0$. Hence for any $\varphi \in S$ and any $\sigma > 0$ there is the neighborhood $U_{\sigma,\varphi}(\varepsilon_0)$ of the point ε_0 such that $U_{\sigma,\varphi}(\varepsilon_0) \subset G^{-1}(O_{\sigma,\varphi}(g))$. The family of neighborhoods $\{U_{\sigma,\varphi}(\varepsilon_0), \sigma > 0, \varphi \in S\}$ is the base of some filter F of subsets of the set E .

Then if F is the ultrafilter majorizing the filter F and μ_F is the corresponding two-valued measure then $g = \int_E G d\mu_F$, therefore $\text{Ls}_{\varepsilon \rightarrow \varepsilon_0} G(\varepsilon) \subset \bigcup_{\mu \in W_0(E)} \int_E G d\mu$. Therefore the equality (1) is proved.

If the limit $\lim_{\varepsilon \rightarrow \varepsilon_0} G(\varepsilon) = g \in Z$ exists then the set $\text{Ls}_{\varepsilon \rightarrow \varepsilon_0} G(\varepsilon)$ is one-point set and therefore $g = \int_E G(\varepsilon) d\mu$ for any $\mu \in W(E)$.

If in addition the set E is arcwise connected and the map G is continuous then the image $G(E)$ is arcwise connected. Therefore for any $\varphi \in S$ the set $\text{Ls}_{\varepsilon \rightarrow \varepsilon_0} \varphi(G(\varepsilon))$ is the segment. According to the equality $W_0(E) = \text{Extr}(W(E))$ any measure $\mu \in W(E)$ is the limit point of convex hull of measures from the set $W_0(E)$. Hence for any $\varphi \in S$ the inclusion $\int_E \varphi(G(\varepsilon)) d\mu \in \text{conv}(\{\int_E \varphi(G(\varepsilon)) d\nu, \nu \in W_0(E)\}) = \text{Ls}_{\varepsilon \rightarrow \varepsilon_0} \varphi(G(\varepsilon))$ holds. Thus

$\bigcup_{\mu \in W(E)} \int_E \varphi(G(\varepsilon)) d\mu \subset \text{Ls}_{\varepsilon \rightarrow \varepsilon_0} \varphi(G(\varepsilon))$ for any $\varphi \in S$ and hence $\bigcup_{\mu \in W(E)} \int_E G(\varepsilon) d\mu \subset \text{Ls}_{\varepsilon \rightarrow \varepsilon_0} G(\varepsilon)$. Conversely according to the first statement of the theorem $\text{Ls}_{\varepsilon \rightarrow \varepsilon_0} G(\varepsilon) = \bigcup_{\mu \in W_0(E)} \int_E G(\varepsilon) d\mu \subset$

$\bigcup_{\mu \in W(E)} \int_E G(\varepsilon) d\mu$.

Remark 1. If the topology $\sigma(Z, S)$ in the space Z is generated by some total family of linear functionals $\mathcal{S} \subset S$ then the limit behavior of the map G as $\varepsilon \rightarrow \varepsilon_0$ is defined by the family of numerical functions $\{f \circ G, f \in \mathcal{S}\}$ on the set E .

3 The semigroups in the Hilbert space $L_2(R^d)$ and corresponding pseudomeasures.

Let D be some domain of finite dimensional (d -dimensional) euclidian space R^d endowing with the Lebesgue measure μ_L . Let $L_2(D)$ be Hilbert space of square-integrable measurable by Lebesgue complex-valued functions on the set D . In this paper we will assume that $D = R^d$. The semigroups of transformations of the space $H = L_2(R^d)$ corresponding to the Cauchy problems for evolutionary equations (Schrödinger equations or heat equations) are considered.

We obtain that any one-parameter semigroup \mathbf{U} of maps of the space $L_2(R^d)$ uniquely defines the pseudomeasure μ_U on the algebra \mathcal{A} which is the minimal algebra containing the cylindrical subsets of Banach space $C(R_+, R^d)$ (see [17, 20]). Here a pseudomeasure on the algebra \mathcal{A} is the additive complex-valued function of set on algebra \mathcal{A} without any assumption on its variation. The linear space of pseudomeasures on the algebra \mathcal{A} is noted by symbol $\mathcal{L}(C(R_+, R^d), \mathcal{A})$.

3.1 The classes of subsets

Let $\mathcal{B}(R^d)$ be the algebra of Borel set in R^d and $\mathcal{B}_b(R^d)$ is the ring of bounded sets in algebra $\mathcal{B}(R^d)$. Suppose that the family $\Pi(R^d)$ of sets of algebra $\mathcal{B}(R^d)$ is defined by the condition: for any element $A \in \Pi(R^d)$ either $A \in \mathcal{B}_b(R^d)$ or $R^d \setminus A \in \mathcal{B}_b(R^d)$. Then the family $\Pi(R^d)$ is algebra.

The set $A_{B_1, \dots, B_m}^{t_1, \dots, t_m}$, $t_1, \dots, t_m \in R_+$, $B_1, \dots, B_m \in \mathcal{B}(R^d)$ in the space $C(R_+, R^d)$ is called cylindrical set iff it is the subset of the space $C(R_+, R^d)$ which is defined by the finite number of conditions

$$A_{B_1, \dots, B_m}^{t_1, \dots, t_m} = \{\xi \in C(R_+, R^d) : \xi(t_j) \in B_j, j \in \{1, \dots, m\}\}. \quad (2)$$

Here $m \in \mathbf{N}$ is arbitrary number and the series of m nondecreasing numbers $t_1, \dots, t_m \in R_+$ is arbitrary; and sets $B_j \in \mathcal{B}(R^d)$ for any $j \in \overline{1, m}$. The finite series of the sets B_j , $j \in \overline{1, m}$ is called by the (m -dimensional) base of cylindrical set (2).

The family of all cylindrical sets with arbitrary base $\{B_j \in \mathcal{B}(R^d), j \in \overline{1, m}\}$ (with bounded base $\{B_j \in \mathcal{B}_b(R^d), j \in \overline{1, m}\}$) is noted as symbol Cyl (is noted as symbol Cyl_b). The symbol Cyl_2 (the symbol $\text{Cyl}_{2,b}$) notes the family of cylindrical sets with two-dimensional base $\{B_j \in \mathcal{B}(R^d), j \in \overline{1, 2}\}$ (the family of cylindrical sets with two-dimensional bounded base $\{B_j \in \mathcal{B}_b(R^d), j \in \overline{1, 2}\}$). Analogously the symbol Cyl_m or $\text{Cyl}_{m,b}$ notes the family of cylindrical sets with m -dimensional base $\{B_j \in \mathcal{B}_b(R^d), j \in \overline{1, m}\}$.

The family \mathcal{P} of sets (2) such that for any $j \in \overline{1, m}$ the set B_j belongs to the algebra $\Pi(R^d)$ is semialgebra of subsets of the space $C(R_+, R^d)$. In fact, the family \mathcal{P} contains the unity $\Omega =$

$C(R_+, R^d)$ and empty set. This family is closed with respect to intersections. For any set $A \in \mathcal{P}$ the complement $\bar{A} \equiv \Omega \setminus A$ is the union of finite number of sets from the family \mathcal{P} . For example if $B_1, B_2 \in \Pi$, $t_1, t_2 \in R_+$ and $A_{B_2}^{t_2}, A_{B_1}^{t_1} \in \mathcal{P}$, then $A_{B_2}^{t_2} \cap A_{B_1}^{t_1} = A_{B_1, B_2}^{t_1, t_2} \in \mathcal{P}$; since $\bar{B}_1, \bar{B}_2 \in \Pi$ then $\bar{A}_{B_1}^{t_1} = A_{\bar{B}_1}^{t_1}$ and $\bar{A}_{B_1, B_2}^{t_1, t_2} = A_{\bar{B}_1, \bar{B}_2}^{t_1, t_2} \cup A_{\bar{B}_1, B_2}^{t_1, t_2} \cup A_{B_1, \bar{B}_2}^{t_1, t_2}$. By induction we obtaine that for any set (2) the following relation holds: $\bar{A}_{B_1, B_2, \dots, B_m}^{t_1, t_2, \dots, t_m} = A_{\bar{B}_1}^{t_1} A_{B_2, \dots, B_m}^{t_2, \dots, t_m} \cup A_{B_1}^{t_1} \bar{A}_{B_2, \dots, B_m}^{t_2, \dots, t_m} \cup A_{\bar{B}_1}^{t_1} \bar{A}_{B_2, \dots, B_m}^{t_2, \dots, t_m}$.

Let \mathcal{A} be the least algebra of sets of the space $C(R_+, R^d)$, containing the semialgebra \mathcal{P} (any element of the algebra \mathcal{A} is the union of finite number of elements of semialgebra \mathcal{P}). Let \mathcal{R}_b be the ring in algebra \mathcal{A} such that for any $S \in \mathcal{R}_b$ the set S is the union of finite number of sets $A_{B_1, \dots, B_m}^{t_1, \dots, t_m}$ of type (2) where $B_j \in \mathcal{B}_b(R^d)$ for any $j \in \overline{1, m}$. Let \mathcal{A}_b be the least subalgebra of algebra \mathcal{A} which containing the ring \mathcal{R}_b . Then $\mathcal{A} = \mathcal{A}_b$ since any element $A \in \mathcal{A}$ can be obtained as the result of finite number of union, complements and intersections of some finite series of sets of the ring \mathcal{R}_b .

3.2 On the generation of pseudomeasure by the semigroup

Any family of continuous linear operators $\mathbf{U}(t)$, $t \in R_+$, in the space H uniquely defines the pseudomeasure $\mu_U \in \mathcal{L}(C(R_+, R^d), \mathcal{A})$, which restriction onto the ring \mathcal{R}_b is given by the equality

$$\mu_U(A_{B_1, \dots, B_m}^{t_1, \dots, t_m}) = (\chi_{B_m}, \mathbf{U}(t_m - t_{m-1}) \mathbf{P}_{B_{m-1}} \mathbf{U}(t_{m-1} - t_{m-2}) \mathbf{P}_{B_{m-2}} \dots \mathbf{U}(t_2 - t_1) \chi_{B_1}). \quad (3)$$

Here \mathbf{P}_B is orthogonal projector in the space H acting as the multiplication on the indicator function of the set B , and χ_B is the indicator function of the set B . Then the equality (3) defines the unique complex-valued additive function on the ring \mathcal{R}_b .

The function $\mu_U : \mathcal{R}_b \rightarrow \mathbf{C}$ can be uniquely continued from the ring \mathcal{R}_b onto the algebra \mathcal{A} by the normalised condition: if $A \in \mathcal{A}$ such that $\Omega \setminus A \in \mathcal{R}_b$ then

$$\mu(A) = 1 - \mu(\Omega \setminus A). \quad (4)$$

The conditions (3), (4) are uniquely defines the additive complex-valued function μ_U on the algebra \mathcal{A} such that $\mu_U(\Omega) = 1$. But the variation of pseudomeasure μ can be infinite.

For example,

1) if $\mathbf{U}(t) = e^{-it\mathbf{L}_\Delta}$, $t \geq 0$ where \mathbf{L}_Δ is Laplace operator in the space R^d then relations (3), (4) define the pseudomeasure $\mu_{U_{-\Delta}}$ which coincides with the Feynman pseudomeasure on the ring \mathcal{R}_b (see [24]). But Feynman pseudomeasure has the infinite variation on the ring \mathcal{R}_b and has no countable additive property.

2) If $\mathbf{U}(t) = e^{t\mathbf{L}_\Delta}$, $t \geq 0$ then the relations (3), (4) define the pseudomeasure μ_{U_Δ} which coincides with the Winer measure on the algebra \mathcal{A} (see [24]). Winer measure is nonnegative normalized countable additive measure on the algebra \mathcal{A} .

3.3 On the pseudomeasures generating semigroups

Any pseudomeasure μ on the algebra \mathcal{A} , any series of increasing real nonnegative numbers t_0, t_1, \dots, t_m and any series of sets $B_1, \dots, B_{m-1} \in \mathcal{B}(R^d)$ define the complex-valued function $\beta_{\mu; B_1, \dots, B_{m-1}}^{t_0, t_1, \dots, t_m}$ on the set $H \times H$. The value of the function $\beta_{\mu; B_1, \dots, B_{m-1}}^{t_0, t_1, \dots, t_m}$ on the ordered pair of indicator function χ_{B_0}, χ_{B_m} of the sets $B_0, B_m \in \mathcal{B}_b(R^d)$ is the complex number $\beta_{\mu; B_1, \dots, B_{m-1}}^{t_0, t_1, \dots, t_m}(\chi_{B_0}, \chi_{B_m}) = \mu(A_{B_0, B_1, \dots, B_m}^{t_0, t_1, \dots, t_m})$. In particular if $m = 1$ then the function $\beta_{\mu}^{t_0, t_1}$ has the value $\beta_{\mu}^{t_0, t_1}(\chi_{B_0}, \chi_{B_1}) = \mu(A_{B_0, B_1}^{t_0, t_1})$ on the ordered pair of indicator functions of the sets $B_0, B_1 \in \mathcal{B}_b(R^d)$.

Then the function $\beta_{\mu; B_1, \dots, B_{m-1}}^{t_0, t_1, \dots, t_m}$ can be extended onto sesquilinear form on the dense linear subspace $H_S \equiv \text{span}\{\chi_B, B \in \mathcal{B}_b(R^d)\}$ of the space $L_2(R^d)$ by the rule of linearity: for any ordered pair of linear combinations of indicator functions $f = \sum_{k=1}^m c_k \chi_{B_{0k}}$ and $g = \sum_{j=1}^n \alpha_j \chi_{B_{1j}}$ the value $\beta_{\mu; B_1, \dots, B_{m-1}}^{t_0, t_1, \dots, t_m}(f, g)$ is defined by the relation

$$\beta_{\mu; B_1, \dots, B_{m-1}}^{t_0, t_1, \dots, t_m}(f, g) = \sum_{k=1}^m \sum_{j=1}^n c_k \bar{\alpha}_j \beta_{\mu; B_1, \dots, B_{m-1}}^{t_0, t_1, \dots, t_m}(\chi_{B_{0k}}, \chi_{B_{1j}}).$$

Definition 1a. The pseudomeasure $\mu \in \mathcal{L}(C(R_+, R^d), \mathcal{A})$ is called continuous if for any increasing system of real nonnegative numbers t_0, t_1, \dots, t_m and any sets $B_1, \dots, B_{m-1} \in \mathcal{B}(R^d)$ the corresponded sesquilinear form $\beta_{\mu; B_1, \dots, B_{m-1}}^{t_0, t_1, \dots, t_m}$ on the space H_S is bounded. I.e. there is the constant $M \geq 0$ such that the following inequality holds

$$\|\beta_{\mu; B_1, \dots, B_{m-1}}^{t_0, t_1, \dots, t_m}\| = \sup_{f, g \in H_S, \|f\|=1, \|g\|=1} |\beta_{\mu; B_1, \dots, B_{m-1}}^{t_0, t_1, \dots, t_m}(f, g)| \leq M. \quad (5a)$$

Definition 1as. The pseudomeasure $\mu \in \mathcal{L}(C(R_+, R^d), \mathcal{A})$ is called continuous uniformly with respect to sets if for any increasing system of real nonnegative numbers t_0, t_1, \dots, t_m the sesquilinear form $\beta_{\mu; B_1, \dots, B_{m-1}}^{t_0, t_1, \dots, t_m}$ on the space H_S is bounded for any collection of the sets $B_1, \dots, B_{m-1} \in \mathcal{B}(R^d)$. I.e. there is the constant $M \geq 0$ such that the inequality (5a) holds for any collection of the sets $B_1, \dots, B_{m-1} \in \mathcal{B}(R^d)$.

Definition 1b. The pseudomeasure $\mu \in \mathcal{L}(C(R_+, R^d), \mathcal{A})$ is called continuous with respect to Lebesgue measure if for any increasing system of real nonnegative numbers t_0, t_1, \dots, t_m there is the constant $M \geq 0$ such that for any sets $B_0, B_1, \dots, B_m \in \mathcal{B}_b(R^d)$ the following inequality holds

$$\mu(A_{B_0, B_1, \dots, B_m}^{t_0, t_1, \dots, t_m}) \leq M^m \prod_{k=0}^m \mu_L(B_k), \quad (5b)$$

where μ_L is Lebesgue measure on R^d .

Definition 2a. The pseudomeasure μ is called uniformly continuous if there is the constant $M \geq 0$ such that for any increasing system of real nonnegative numbers t_0, t_1, \dots, t_m and any sets $B_1, \dots, B_{m-1} \in \mathcal{B}(R^d)$ the inequality (5a) holds.

Definition 2b. The pseudomeasure μ is called uniformly continuous with respect to Lebesgue measure if there is the constant $M \geq 0$ such that for any increasing system of real nonnegative numbers t_0, t_1, \dots, t_m and any sets $B_0, B_1, \dots, B_m \in \mathcal{B}_b(R^d)$ the inequality (5b) holds.

The condition of the definition 1a is the stronger than the condition of definition 1a, but any of it is no stronger and no weaker than the condition of the definition 1b. Let us consider the important example of Feynman pseudomeasure μ_F which is generated by the unitary group $\mathbf{U}(t) = e^{i\Delta t}$, $t \in R$, according to the equality (3). This pseudomeasure μ_F satisfies the conditions of the definition 2a with the constant $M = 1$ in the inequality (5a). The conditions of the definition 1b are hold for any increasing system of real nonnegative numbers t_0, t_1, \dots, t_m but the constant M in the inequality 1b depends on this system of numbers t_0, t_1, \dots, t_m and the conditions of the definition 2b are not hold.

If the pseudomeasure μ is continuous (in the sense of definition 1a) then the function $\beta_\mu^{t_0, t_1}$ can be uniquely continued onto the sesquilinear function on the space $L_2(R^d)$. Analogously for any increasing system of real nonnegative numbers t_0, t_1, \dots, t_m and any sets $B_1, \dots, B_{m-1} \in \mathcal{B}(R^d)$ the function $\beta_{\mu; B_1, \dots, B_{m-1}}^{t_0, t_1, \dots, t_m}$ has the unique continuation onto the continuous sesquilinear functional on the space $L_2(R^d)$.

The continuous sesquilinear functional $\beta_{\mu; B_1, \dots, B_{m-1}}^{t_0, t_1, \dots, t_m}$ uniquely defines the continuous linear operator $\mathbf{A}_{\mu; B_1, \dots, B_{m-1}}^{t_0, t_1, \dots, t_m} \in B(L_2(R^d))$ such that

$$\beta_{\mu; B_1, \dots, B_{m-1}}^{t_0, t_1, \dots, t_m}(u, v) = (\mathbf{A}_{\mu; B_1, \dots, B_{m-1}}^{t_0, t_1, \dots, t_m} u, v)_{L_2} \quad \forall u, v \in L_2(R^d). \quad (6)$$

In particular if $m = 1$ then the continuous pseudomeasure μ defines the two-parametric family of operators $\mathbf{A}_\mu^{t_0, t_1}$, $t_0, t_1 \in R_+$.

Definition 3. The continuous pseudomeasure μ is called Markovian if the family of operators $\mathbf{A}_{\mu; B_1, \dots, B_{m-1}}^{t_0, t_1, \dots, t_m}$, $t_0, t_1, \dots, t_m \in R_+$, $B_1, \dots, B_{m-1} \in \mathcal{B}(R^d)$, satisfies the conditions:

1) Markov condition (evolutionary condition):

$$\mathbf{A}_{\mu; B_1, \dots, B_{m-1}}^{t_0, t_1, \dots, t_m} \mathbf{P}_{B_m} \mathbf{A}_{\mu; B_{m+1}, \dots, B_{m+p-1}}^{t_m, t_{m+1}, \dots, t_{m+p}} = \mathbf{A}_{\mu; B_1, \dots, B_{m+p-1}}^{t_0, \dots, t_{m+p}} \\ \forall t_0 \leq t_1 \leq \dots \leq t_{m+p} \in R_+ \quad \forall B_1, \dots, B_{m+p-1} \in \mathcal{B}(R^d); \quad (7)$$

2) the normalise condition: $\mathbf{A}_\mu^{t_0, t_0} = \mathbf{I}$ for all $t_0 \in R_+$.

If the pseudomeasure μ is continuous and Markovian then it defines by means of equality

(6) the two-parametric family of operators $\mathbf{A}_\mu^{t_1, t_2}$, $t_1, t_2 \in R$, which satisfies the equality:

$$\mathbf{A}_\mu^{t_1, t_2} \mathbf{A}_\mu^{t_2, t_3} = \mathbf{A}_\mu^{t_1, t_3} \quad \forall t_1, t_2, t_3 \in R_+ : t_1 \leq t_2 \leq t_3. \quad (8)$$

Remark 2. Markovian property of continuous pseudomeasure μ gives the opportunity to define the pseudomeasure μ on the algebra \mathcal{A} by its restriction onto the class of subsets Cyl_2 .

Definition 4. The pseudomeasure μ is called stationary if it is invariant with respect to the shift of all time parameters on some real variable:

$$\mu(A_{B_0, \dots, B_m}^{t_0, \dots, t_m}) = \mu(A_{B_0, \dots, B_m}^{t_0+s, \dots, t_m+s}) \quad (9)$$

for any $s \in R_+$, $A_{B_0, \dots, B_m}^{t_0, \dots, t_m} \in \text{Cyl}$.

Remark 3. If the Markovian pseudomeasure μ on the algebra \mathcal{A} is stationary then the equalities (6) defines the two-parametric family of operators $\mathbf{A}_\mu^{t_1, t_2}$ such that it depends only on the difference $t_2 - t_1$:

$$\mathbf{A}_\mu^{s, t+s} = \mathbf{V}_\mu(t), \quad t \in R_+, \quad \forall s \in R_+. \quad (10)$$

Then the pseudomeasure μ defines the one-parametric semigroup $\mathbf{V}_\mu(t)$, $t \in R_+$ in the space H . The presentation and the approximation of the semigroup generating by the Cauchy problem for the evolution equation can be given by the means of Feynman formulas (see [4]).

Remark 4. If the pseudomeasure μ is Markovian but is not stationary then the equality (6) defines the two-parametric family of bounded linear operators $\mathbf{A}_\mu^{s, t}$, $0 \leq s \leq t < +\infty$, which is called evolutionary family. This evolutionary family describes the dynamics of a solution of Cauchy problem with time depended Hamiltonian.

Remark 5. If continuous pseudomeasure is not markovian then the family of operators $\mathbf{U}_\mu^{s, t}$, $0 \leq s \leq t < +\infty$, has no evolutionary property (8). According to the works [16], [18] any limit point for the sequence of regularised semigroups in the space of quantum states is the family of dynamical maps which has no evolutionary property.

Remark 6. If the pseudomeasure μ defines the diffusion random process with value in the space R^d , then the generator the semigroup \mathbf{U}_μ obtained by the equation (3) is linear second order elliptic differential operator.

Proposition 1. *If the pseudomeasure μ_U is generated by the group of unitary operators $\mathbf{U}(t)$, $t \in R$, according to equality (3) then the pseudomeasure μ_U possesses Markovian, continuity and stationarity properties.*

In fact, for all $T > 0$ the following estimate holds

$$\sup_{|t| < T, v, u \in S_2, \|v\|_H=1, \|u\|_H=1} |(u, \mathbf{V}_\mu(t)v)| \leq 1.$$

Therefore pseudomeasure μ_U satisfy the continuity condition (5) with the constant $M = 1$. According to the equality (3) the Markovian property and stationarity property of pseudomeasure μ_U is the consequence of semigroup properties of the family of operators $\mathbf{U}(t)$, $t \in R$.

Corollary 1. If the pseudomeasure μ_U is generated by the semigroup $\mathbf{U}(t)$, $t \in R$ according to the equality (3) then the semigroup can be uniquely reconstructed by pseudomeasure μ_U according to the equality (10). Conversely if the semigroup \mathbf{V}_μ is defined by the continuous

Markovian stationary pseudomeasure μ according to the equalities (10), then the semigroup \mathbf{V}_μ uniquely defines the pseudomeasure μ by the equalities (3).

Therefore we can identify the one-parametes semigroups of operators and the stationary Marcovian continuous pseudomeasures. Hence the passage to the limit for the sequences of solutions of the Cauchy problems in the space of solutions $C(R_+, L_2(R^d))$ is corresponded to the passage to the limit for the sequences of pseudomeasures on the space $C(R_+, R^d)$ in the space of pseudomeasures $\mathcal{L}(C(R_+, R^d), \mathcal{A})$.

Let symbol $\mathcal{L}_{cont}(C(R_+, R^d), \mathcal{A})$ (or symbol $\mathcal{L}_{ucont}(C(R_+, R^d), \mathcal{A})$) note the set of pseudomeasures in the space $\mathcal{L}(C(R_+, R^d), \mathcal{A})$ which is continuous (or uniformly continuous). Both of this sets are the linear subspaces in the space $\mathcal{L}(C(R_+, R^d), \mathcal{A})$.

3.4 On the topologies on the pseudomesures space.

The linear space $\mathcal{L}(C(R_+, R^d), \mathcal{A})$ can be endowed with the different topologies. The article [23] is devoted to the description of wide class of this topologies, to the investigation of compantness and convergence for the sequences of Radon measures on the completely regular topological spaces, to investigation of the properties of limit measures.

Let us introduce the equivalence relation on the space of pseudomeasures $\mathcal{L}(C(R_+, R^d), \mathcal{A})$. The pseudomeasure $\mu \in \mathcal{L}(C(R_+, R^d), \mathcal{A})$ is called equivalent to the pseudomeasure $\nu \in \mathcal{L}(C(R_+, R^d), \mathcal{A})$ if the equality $\mu(A) = \nu(A)$ holds for any set $A \in \text{Cyl}_b$. For examle if the pseudomeasure μ is defined by the unitary group \mathbf{U} according to the equalities (3), (4), and pseudomeasure ν is defined by the same group \mathbf{U} according to the equality (3) and the equality $\mu_{\mathbf{U}}(A) = -\mu_{\mathbf{U}}(\Omega \setminus A)$, $\Omega \setminus A \in \mathcal{R}_b$ instead of the condition (4) then any of this two pseudomeasures are equivalent to each other.

Let symbol \mathcal{L}_{ucL} note the set of uniformly continuous with respect to Lebesgue measure pseudomeasures on the algebra \mathcal{A} (see definitions 1b, 2b). Then the set \mathcal{L}_{ucL} is the linear subspaces in the space $\mathcal{L}(C(R_+, R^d), \mathcal{A})$.

Lemma 1. *Suppose that the map $p : \mathcal{L}_{uc} \rightarrow [0, +\infty)$ has value $p(\nu) = m_\nu$ on arbitrary pseudomeasure $\nu \in \mathcal{L}_{uc}$ which is equal to the least constant M_ν in the inequality (5b) (see definition 2b). Then the functional p is the norm on the linear space \mathcal{L}_{ucL} .*

In fact, the functional p is defined on the space \mathcal{L}_u and has value $p(\nu) \in [0, +\infty)$ at any point $\nu \in \mathcal{L}_{uc}$. The functional p on the space \mathcal{L}_{uc} is obviously nonnegative and uniform. The triangle inequality $m_{\nu_1+\nu_2} \leq m_{\nu_1} + m_{\nu_2}$ is the concequence of the definition 1b and the definition of summ of pseudomeasures. If $p(\mu) = m_\mu = 0$ for some pseudomeasure $\mu \in \mathcal{L}_{uc}$ then according to the definition 1b $\mu(A) = 0$ for any $A \in \text{Cyl}_b$, hence $\mu = \theta_{\mathcal{L}_{ucL}}$.

Theorem 2. *Normalised linear space (\mathcal{L}_{ucL}, p) is Banach space.*

Let $\{\nu_k\}$ be fundamental sequence of pseudomesures in the normalised space (\mathcal{L}_{ucL}, p) :

$\sup_{n \in \mathbf{N}} p(\nu_k - \nu_{k+n}) \rightarrow 0$ as $k \rightarrow \infty$. Therefore for any $A_{B_0, \dots, B_m}^{t_0, \dots, t_m} \in \text{Cyl}_{b, m+1} \subset \text{Cyl}_b \subset \mathcal{A}$ the inequality $|\nu_k(A) - \nu_{k+n}(A)| \leq p(\nu_k - \nu_{k+n})^m \prod_{j=0}^m \mu_L(B_j)$ holds. Then according to definition 1b the numerical sequence $\{\nu_k(A)\}$ is fundamental for any $A \in \mathcal{A}$. Hence the equality

$$\nu(A) = \lim_{n \rightarrow \infty} \nu_n(A) \quad \forall A \in \mathcal{A} \quad (11)$$

uniquely defines the function $\nu(A)$, $A \in \mathcal{A}$ on the algebra \mathcal{A} .

Since the function ν_n on the algebra \mathcal{A} is additive for any $n \in \mathbf{N}$, then according to (11) the function ν on algebra \mathcal{A} is additive and hence $\nu \in \mathcal{L}(C(R_+, R^d), \mathcal{A})$. Let us prove that $\nu \in \mathcal{L}_{ucL}(C(R_+, R^d), \mathcal{A})$. For any $n \in \mathbf{N}$ there is the constant $M_n = p(\nu_n)$ such that the inequality $|\nu(A)| \leq M_n^m \prod_{j=0}^m \mu_L(B_j)$ holds for any $A_{B_0, \dots, B_m}^{t_0, \dots, t_m} \in \text{Cyl}_{b, m+1} \subset \text{Cyl}_b \subset \mathcal{A}$. The sequence $\{M_n\}$ is bounded according to the fundamentality of the sequence $\{\nu_n\}$ in the space (\mathcal{L}_{ucL}, p) . Therefore there is the constant $M \geq 0$ such that the inequality $|\nu(A)| \leq M^m \prod_{j=0}^m \mu_L(B_j)$ holds for any $A_{B_0, \dots, B_m}^{t_0, \dots, t_m} \in \text{Cyl}_{b, m+1} \subset \text{Cyl}_b \subset \mathcal{A}$. Theorem 2 is proved.

Further some topologies on the linear space $\mathcal{L}(C(R_+, R^d), \mathcal{A})$ will be introduced. Let symbol σ_{cyl} note the topology on the linear space $\mathcal{L}(C(R_+, R^d), \mathcal{A})$ which is defined by the family of cylindrical linear functionals

$$\{P_A, A \in \mathcal{A}\}, \quad (12)$$

where $P_A(\mu) = \mu(A)$.

Let $\sigma_{cyl, 2}$ be the topology on the linear space $\mathcal{L}(C(R_+, R^d), \mathcal{A})$ which is defined by the linear functionals $P_A, A \in \text{Cyl}_2$. It is easy to show that the topology $\sigma_{cyl, 2}$ is more weak than the topology σ_{cyl} .

Let H_S be the pre-Hilbert space of step complex-valued functions on the space R^d (i.e. the linear hull of the set of indicator functions for borel subsets $B \in \mathcal{B}(R^d)$) which is endowed with the scalar product of the space H (see definition 1a).

For any pair of functions $v, w \in H_S$ and any pair of numbers $s, t > 0$, $s < t$, the functional $P_{s, t, v, w}$ on the pseudomeasures space $\mathcal{L}(C(R_+, R^d), \mathcal{A})$ is defined by the rule:

if the functions v, w are indicator functions of Borel subsets $V, W \in \mathcal{B}(R^d)$ respectively then the value $P_{s, t, v, w}(\mu)$, $\mu \in \mathcal{L}$, is equal to the value $\mu(A_{V, W}^{s, t})$ of pseudomeasure μ on the cylindrical set with two-dimensional base $A_{V, W}^{s, t} = \{\xi : \xi(s) \in V, \xi(t) \in W\} \subset \mathcal{A}$.

for linear combination of indicator function the the value of the functional $P_{s, t, v, w}(\mu)$ on the pseudomeasure μ is obtained by linearity.

Then for any Markovian continuous pseudomeasure μ the following equality holds

$$P_{s, t, v, w}(\mu) = \left| \int_{C(R_+, R)} (v(\xi(s)), w(\xi(t))) d\mu \right| = |(w, \mathbf{U}(t-s)v)| = \mu(A_{V, W}^{s, t}). \quad (13)$$

Hence the topology on the linear space $\mathcal{L}(C(R_+, R^d), \mathcal{A})$ which is defined by the set of functionals (13) coincides with the topology $\sigma_{cyl,2}$ which is generated by cylindrical sets with two-dimensional bases. Therefore it is more weak than the topology which is generated by all cylindrical subsets.

To the investigation of uniform convergence of the sequence of pseudomeasures we introduce the topology $\Sigma_{cyl,2}$ on the linear space $\mathcal{L}(C(R_+, R^d), \mathcal{A}_b)$ which is generated by the system of nonlinear functionals $\{P_{v,T}, T > 0, v \in H\}$ where each seminorm $P_{v,T}$ is given by the formula:

$$P_{v,T}(\mu) = \sup_{t \in [0,T], w \in H_S, \|w\|=1} P_{0,t,v,w}(\mu); \quad \mu \in \mathcal{L}(C(R_+, R^d), \mathcal{A}). \quad (14)$$

The topology on the linear space $\mathcal{L}(C(R_+, R^d), \mathcal{A})$ which is defined by the system of functionals $P_{s,t,v,w}$, $s, t > 0, v, w \in H$, is corresponded to the topology on the space $C_s(R_+, B(H))$ of strongly continuous operator-valued functions which is generated by the system of functionals $p_{s,t,v,w}$, $s, t > 0, v, w \in H$, in the following sense: for any pseudomeasure $\mu \in \mathcal{L}(C(R_+, R^d), \mathcal{A})$ the operator-function $\mathbf{U}_\mu(s, t)$, $s, t > 0$ arising according to the equality (10) satisfy the equalities

$$P_{s,t,v,w}(\mu) = \left| \int_{C(R_+, R^d)} (v(\xi(s)), w(\xi(t))) d\mu \right| = |(w, \mathbf{U}_\mu(s, t)v)| = p_{s,t,v,w}(\mathbf{U}_\mu(t))$$

for any pair of numbers $s, t > 0$ and any pair of elements $v, w \in H$.

Analogously for any strongly continuous semigroup $\mathbf{U} \in C_s(R_+, B(H))$ the pseudomeasure $\mu \in \mathcal{L}(C(R_+, R^d), \mathcal{A})$ arising according to the equality (3) (or if the stationary Markovian continuous pseudomeasure μ defines the semigroup \mathbf{U} according to equality (10)), satisfy the equalities

$$P_{T,v}(\mu) = p_{T,v}(\mathbf{U}), \quad T > 0, v \in H. \quad (15)$$

The following statement is the consequence of the equalities (15). The formulation of this statement and the scheme of the proof are published in the paper [17]. Now we give the proof with details of this statement.

Theorem 3. *Let $\{\mu_n\}$ be the sequence of Markovian pseudomeasures on the algebra \mathcal{A} such that for any $n \in \mathbf{N}$ the pseudomeasure μ_n is uniformly continuous. Suppose that the pseudomeasure μ_n defines the unitary semigroup $\mathbf{U}_n(t)$, $t \in R_+$, in the space $H = L_2(R^d)$ according to the equality (10). Any semigroup $\mathbf{U}_n(t)$, $t \in R_+$, has the self-adjoint generators \mathbf{L}_n .*

Then the convergence of the sequence of semigroups in the strong operator topology of the space $B(H)$ uniformly on arbitrary segment of semiaxis R_+ to the limit operator-valued function \mathbf{F} is equivalent to the convergence of the sequence of pseudomeasures μ_n to the limit function of set ν on the algebra \mathcal{A} in the topology $\Sigma_{cyl,2}$.

If one of two equivalent conditions is fulfilled then

1. The limit operator-function \mathbf{F} is semigroup.
2. The sequence μ_n converges to the limit function of a set $\nu \in \mathcal{L}$ in the topology σ_{cyl} . The limit function of a set ν is Markovian, uniformly continuous and stationary pseudomeasure.
3. The limit pseudomeasure ν defines the semigroup $\mathbf{F}(t)$ according to the equality (10). The limit pseudomeasure ν can be defined by the semigroup $\mathbf{F}(t)$ according to the equalities

$$\nu(A) = \lim_{n \rightarrow \infty} \mu_n(A) = (\chi_{B_m}, \mathbf{F}(t_m - t_{m-1}) \mathbf{P}_{B_{m-1}} \mathbf{F}(t_{m-1} - t_{m-2}) \dots \mathbf{P}_{B_2} \mathbf{F}(t_2 - t_1) \chi_{B_1}), \quad (16)$$

which satisfy for arbitrary set $A = A_{B_1, \dots, B_m}^{t_1, \dots, t_m} \in \mathcal{A}_f$.

Proof. According to (15) for any $n, k \in \mathbf{N}$ $T > 0$, $v \in H$ the equality $P_{T,v}(\mu_n - \mu_k) = p_{T,v}(U_n - U_k)$ holds. Therefore if the sequence of pseudomeasures $\{\mu_n\}$ converges in the topology generated by the functionals (14) to the limit function of a set ν on the class of subsets Cyl_2 , then the semigroup sequence $\{\mathbf{U}_n\}$ converges in the strong operator topology uniformly on arbitrary segment to the limit operator-function $\mathbf{U}(t)$. The limit operator function is semigroup (see [12], [17]). Hence the equality $(\chi_{B_0}, \mathbf{F}(t) \chi_{B_1}) = \nu(A_{B_0, B_1}^{0,t})$ is the consequence of the sequence of equalities $(\chi_{B_0}, \mathbf{U}_n(t) \chi_{B_1}) = \mu_n(A_{B_0, B_1}^{0,t})$, $n \in \mathbf{N}$. Therefore the semigroup \mathbf{F} is generated by the limit pseudomeasure ν in accordance with the equality (10).

Conversely if the sequence of unitary semigroups $\{\mathbf{U}_n\}$ converges in the strong operator topology uniformly on arbitrary segment to the limit operator function $\mathbf{U}(t)$, $t \in R_+$, then the limit function \mathbf{U} is isometric semigroup and the corresponding sequence of pseudomeasures $\{\mu_n\}$ converges in the topology generated by the functionals (14) to the limit function of a set $\nu(A)$, $A \in Cyl_2$.

Let us prove that if the sequence of continuous Markovian pseudomesures converges in the topology Σ_{cyl2} to the function of a set ν on the class of subsets Cyl_2 then it converges in the topology generated by the functionals of a class Cyl_b to the limit Markovian pseudomeasure \mathcal{V} on the algebra \mathcal{A} such that its restrictions on the class $Cyl_{2,b}$ coincides with the function ν and moreover the limit pseudomesure \mathcal{V} and limit semigroup \mathbf{F} satisfy the equalities (10) and (16).

According to the theorem assumption any pseudomeasure μ_n , $n \in \mathbf{N}$, is Markovian (see definition 3 and equality (7)), therefore for any $t, s > 0$ and any $V, W, Y \in \mathcal{B}_f$ the following equalities

$$\mu_n(A_{V,W,Y}^{0,t,t+s}) = (\chi_Y, \mathbf{U}_n(s) \mathbf{P}_W \mathbf{U}_n(t) \chi_V) = (\mathbf{U}_n(-s) \chi_Y, \mathbf{P}_W \mathbf{U}_n(t) \chi_V) \quad (17)$$

holds. To obtaining the last equation we use the equality $(\mathbf{U}_n(s))^* = \mathbf{U}_n(-s)$, $s \in R_+$, $n \in \mathbf{N}$, (where $\mathbf{U}_n(s) = e^{-is\mathbf{L}_n}$) which is the consequence of self-adjointness of operators \mathbf{L}_n , $n \in \mathbf{N}$ with arbitrary $s \geq 0$.

According to the first statement of the theorem the convergence of semigroup sequence $\mathbf{U}_n(t)$ in the strong operator topology uniformly on any segment is equivalent to convergence of the

sequence $\{\mu_n\}$ of continuous Markovian pseudomesures in the topology Σ_{cyl2} to the limit function of a set on the class Cyl_2 . Then according to the convergence of semigroup sequence $\mathbf{U}_n(t)$ in the strong operator topology uniformly on any segment the following statement holds. The sequences of H -valued functions $\{\mathbf{U}_n(t)\chi_V\}$ and $\{\mathbf{U}_n(-s)\chi_Y\}$ converges uniformly on arbitrary segment in the space $L_2(R^d) = H$ to the functions $\mathbf{U}(t)\chi_V$ and $\mathbf{U}(-s)\chi_Y$ respectively. Therefore the limit function of a set ν satisfies the equalities $\nu(A_{V,W,Y}^{0,t,t+s}) = (\chi_Y, \mathbf{U}(s)\mathbf{P}_W\mathbf{U}(t)\chi_V)$. Hence it satisfies the equality (7) for arbitrary pair of cylindrical subsets with two-dimensional base.

For any $n \in \mathbf{N}$ the pseudomeasure μ_n is generated by the semigroup $\mathbf{U}_n(t)$, $t \geq 0$, according to the equality (3). Therefore for arbitrary increasing system of nonnegative numbers t_0, t_1, \dots, t_m and any system of Borel sets $B_1, \dots, B_{m-1} \in \mathcal{B}_b(R^d)$ the pseudomeasure μ_n defines the sesquilinear forms $\beta_{\mu_n; B_1, \dots, B_{m-1}}^{t_0, t_1, \dots, t_m}$ and bounded operators (6). Since the pseudomeasure μ_n satisfies the equality (3) then the operator (6) admits the following presentation $\mathbf{A}_{\mu_n; B_1, \dots, B_{m-1}}^{t_0, t_1, \dots, t_m} = \mathbf{U}_n(t_1 - t_0)\mathbf{P}_{B_1} \dots \mathbf{P}_{B_{m-1}}\mathbf{U}_n(t_m - t_{m-1})$. Then according to the convergence of semigroup sequence $\{\mathbf{U}_n\}$ in the strong operator topology uniformly on arbitrary segment the sequence of operators $\{\mathbf{A}_{\mu_n; B_1, \dots, B_{m-1}}^{t_0, t_1, \dots, t_m}\}$ converges in the strong operator topology to the limit operator $\mathbf{A}_{B_1, \dots, B_{m-1}}^{t_0, t_1, \dots, t_m} = \mathbf{F}(t_1 - t_0)\mathbf{P}_{B_1} \dots \mathbf{P}_{B_{m-1}}\mathbf{F}(t_m - t_{m-1})$ for arbitrary increasing system of nonnegative numbers t_0, t_1, \dots, t_m and any system of Borel sets $B_1, \dots, B_{m-1} \in \mathcal{B}_b(R^d)$. Hence the sequence of functions $\{\mu_n\}$ on the ring \mathcal{R}_b converges point-wise on the ring \mathcal{R}_b to the function of set \mathcal{V} such that the equality

$$\mathcal{V}(A_{B_0, B_1, \dots, B_m}^{t_0, t_1, \dots, t_m}) = (\chi_{B_0}, \mathbf{F}(t_1 - t_0)\mathbf{P}_{B_1} \dots \mathbf{P}_{B_{m-1}}\mathbf{F}(t_m - t_{m-1})\chi_{B_m}) \quad (18)$$

holds for any increasing system of nonnegative numbers t_0, t_1, \dots, t_m and any system of Borel sets $B_1, \dots, B_{m-1} \in \mathcal{B}_b(R^d)$. Hence the function of a set \mathcal{V} is the continuation of the function ν from the class Cyl_2 onto the ring \mathcal{R}_b and has the unique continuation onto the measure μ_F on the algebra \mathcal{A} . Thus according to (18) the limit pseudomeasure μ_F is defined by the limit semigroup \mathbf{F} in accordance with the equality (3), (4) and the equality (16) is fulfilled, i.e. $\mathcal{V} = \mu_F$.

If μ_F is stationary Markovian pseudomeasure on the algebra \mathcal{A} which is defined by the semigroup \mathbf{F} in accordance with the rule (17) then the pseudomeasure μ_F satisfies the uniform continuity condition (5a) with the constant $M = 1$ due to the unitarity of semigroup \mathbf{F} . Moreover for any $T > 0$, $v \in H$ the equality $\lim_{n \rightarrow \infty} P_{T,v}(\mu_n - \mu_F) = 0$ holds.

Since any of semigroups generating by the pseudomeasures μ_n is unitary then any of the constants M^n in the conditions (5a) of pseudomeasure μ_n continuation can be equal to unity. Since the limit pseudomeasure ν is defined by the limit semigroup \mathbf{F} in accordance with the equalities (18) then pseudomeasure ν is also uniformly continuous with the constant $M = 1$. Hence the limit pseudomeasure ν generates the semigroup \mathbf{F} according to the equality (3) and can be obtained by using of semigroup \mathbf{F} in accordance with the equality (16). Theorem 3 is proved.

Corrolary 2. The set $\mathcal{L}_{sg}(C(R_+, R), \mathcal{A})$ of pseudomeasures generating a semigroups is closed in the topology Σ_{Cyl_2} .

But the convergence of semigroups sequence in more weak topology can not give the information about the convergence of corresponding pseudomeasures sequence on the hole algebra \mathcal{A} . Let us note by symbol $\mathcal{M}_a(C(R_+, R^d), Cyl_2)$ the linear space of additive functions on the class of subsets Cyl_2 (which is not closed with respect to finite number of intersection and unity operations). Let $\mathcal{W} = \{P_{0,t,v,w}, t > 0, v, w \in H\}$ be the set of linear functionals on the linear space $\mathcal{M}_a(C(R_+, R^d), Cyl_2)$ acting by the formula (13) (see [20]). Let $\tau_{\mathcal{W}}$ be the topology on the space $\mathcal{M}_a(C(R_+, R^d), Cyl_2)$ which generated by the set of functionals \mathcal{W} . Then the convergence of pseudomeasures sequence in the topology $\tau_{\mathcal{W}}$ is equivalent to the convergence of corresponding operator-valued functions sequence in the weak operator topology (see [20]).

Theorem 4. Let $\{\mathbf{U}_n(t), t \geq 0\}$ be the sequence of unitary semigroups acting in the space $H = L_2(R^d)$, and $\{\mu^{\mathbf{U}_n}\}$ is the sequence of additive functions on the class $Cyl_{2,b}$ generated by the semigroups $\mathbf{U}_n(t), t \geq 0$ according to the equality (3).

Then the point-wise convergence on the semiaxe R_+ of the sequence of semigroups $\{\mathbf{U}_n(t), t > 0\}$ in the weak operator topology to the limit operator-valued function $\mathbf{F}(t)$ is equivalent to the convergence of functions of the set sequence $\{\mu_n\}$ in the topology $\tau_{\mathcal{W}}$ to the function of the set $\nu \in \mathcal{M}_a(C(R_+, R^d), Cyl_2)$ which is defined by the operator-valued function \mathbf{F} by the equalities

$$\nu(A) = \lim_{n \rightarrow \infty} \mu_n(A) = (\chi_{B_2}, \mathbf{F}(t_2 - t_1)\chi_{B_1}), \quad \forall A = A_{B_1, B_2}^{t_1, t_2} \in Cyl_2. \quad (19)$$

In fact if the sequence of the semigroups $\{\mathbf{U}_n\}$ converges to the limit operator valued function $\mathbf{F}(t), t > 0$, in the weak operator topology pointwise on the semiaxe R_+ , then the sequence of pseudomeasures $\{\mu^{\mathbf{U}_n}\}$ converges in the topology $\tau_{\mathcal{W}}$ to the limit function of a set $\nu(A), A \in Cyl_2$ which is defined by the limit operator valued function $\mathbf{F}(t), t > 0$, according to the equality (19).

Conversely, suppose that the sequence of the pseudomeasures $\{\mu_n\}$ is defined by the sequence of strongly continuous semigroups \mathbf{U}_n by the equalities (3). If the sequence $\{\mu_n\}$ converges in the topology $\tau_{\mathcal{W}}$ to the limit function of a set $\nu \in \mathcal{M}_a(C(R_+, R), Cyl_2)$ on the class of the sets Cyl_2 then the sequence of the semigroups $\{\mathbf{U}_n\}$ converges in the weak operator topology pointwise on the semiaxe R_+ to the limit operator valued function $\mathbf{F}(t)$ which is weakly continuous and satisfy the equality (19). Theorem 4 is proved.

The function of a set ν is defined on the class Cyl_2 only since the convergence of semigroups sequence can't obtain the limit behavior of sequence of the values of pseudomeasures μ_n on the arbitrary sets of the algebra \mathcal{A} . In addition if the semigroup sequence $\{\mathbf{U}_n\}$ converges in the

weak operator topology uniformly on arbitrary segment only then the equality (17) should not be fulfilled.

In fact, if $A_{V,W,Y}^{0,t,t+s} \in Cyl_3 \subset \mathcal{A}$ is the cylindrical set with three dimensional base then for any $n \in \mathbb{N}$ the Markovian pseudomeasures μ_n satisfy the equality (17). But according to the work [16] (or [12]) the following inequality $\lim_{n \rightarrow \infty} (\mathbf{U}_n(s)\chi_Y, \mathbf{P}_W \mathbf{U}_n(t)\chi_V) \neq (\mathbf{F}(s)\chi_Y, \mathbf{P}_W \mathbf{F}(t)\chi_V)$ holds. Moreover, the numerical sequence $\{(\mathbf{U}_n(s)\chi_Y, \mathbf{P}_W \mathbf{U}_n(t)\chi_V)\}$ can diverges fore some choice of the sets $Y, V, W \in \mathcal{B}_f(R^d)$.

4 Random semigroups

. The following extension of notion of random variable is introduced. *Random variable is defined as the measurable map of the space with finite additive measure $(\Omega, \mathcal{F}, \mu)$ into the measurable space (Y, \mathcal{A}) .* The random variable with the value in topological space (Z, τ) is defined as the measurable map into the measurable space (Z, \mathcal{A}_τ) where algebra of (Borel) subsets \mathcal{A}_τ is the least algebra containing the topology τ (i.e. containing any open subset of the space (Z, τ)).

Let E be some topological space and X is some Banach space which has the predual space X_* . The set E can be presented by the set $G(X)$ of all generators of strongly continuous semigroups of transformations of Banach space X which is endowed with the topology of strong (or weak) graph convergence.

According to our notations the symbol $W(E)$ notes the set of nonnegative normalized finite additive measures on the measurable space $(E, 2^E)$. For any $\varepsilon_0 \in E$ the symbol $W(E, \varepsilon_0)$ notes the set of measures $\{\mu \in W(E) \text{ such that } 1) \mu(A) = 0 \text{ if the closure } \bar{A} \text{ of the set } A \subset E \text{ does not contain the point } \varepsilon_0; 2) \mu(\{\varepsilon_0\}) = 0.$

Let $(E, 2^E, \mu)$ be the measurable space endowing with the measure $\mu \in W(E)$. Let ξ is random variable which is defined on the set E and takes values in the topological space $Z_{s(w)} = C_{s(w)}(R_+, B(X))$ of strongly (weakly) continuous maps of semiaxe R_+ into the Banach space $B(X)$ of bounded linear operators in the space X .

Let us consider a family of functionals $\{\varphi_{t,A,g}, t \in R_+, A \in X, g \in X_*\}$ on the space Z_w such that $\varphi_{t,A,g}(z) = \sup_{\tau \in [0,t]} \langle z(\tau)A, g \rangle$, $t \in R_+, A \in X, g \in X_*$ for any $z \in Z_w$. Let τ_w be the topology on the linear space Z_w which is generated by the family of functionals $\varphi_{t,A,g}, t \in R_+, A \in X, g \in X_*$.

The space Z_s can be endowed with the family of the functionals $\{\Phi_{t,A}, t \in R_+, A \in X\}$ such that $\Phi_{t,A}(z) = \sup_{\tau \in [0,t]} \sup_{g \in X_*, \|g\|_{X_*}=1} \langle z(\tau)A, g \rangle$, $t \in R_+, A \in X$ for any $z \in Z_s$. This family of functionals generates the topology τ_s on the space Z_s .

Then topological vector space Z (Z_s or Z_w) endowing with the algebra of Borel subsets \mathcal{A}_τ is the measurable space and the map $\xi : E \rightarrow Z$ is measurable. Therefore ξ is random variable.

A random variable $\xi : E \rightarrow Z_s$ ($\xi : E \rightarrow Z_w$) is called random semigroup iff its values $\xi(\varepsilon)$, $\varepsilon \in E$, are strong (weak) continuous semigroups.

Then the linear space Z (Z_s or Z_w) endowing the structure of algebra of Borel subsets is the measurable space and the map $\xi : E \rightarrow Z$ is the random variable.

The mean value of random variable ξ as the map of the space with the measure $(E, 2^E, \mu)$ into the linear topological space Z is Pettis integral

$$M\xi = \int_E \xi_\varepsilon d\mu(\varepsilon),$$

where $M\xi$ is the element of the space Z such that the equality

$$\langle M\xi(t)A, g \rangle = \int_E \langle \xi_\varepsilon(t)A, g \rangle d\mu(\varepsilon) \quad (20)$$

holds for any $t \in R_+$, $A \in X$, $g \in X_*$. Here the integral in right hand side of equality (20) is Radon integral of bounded measurable complex-valued function on the set E by the finite additive measure μ .

Lemma 2. For any $t \in R_+$ the family of equalities (20) with arbitrary $A \in X$, $g \in X_*$ uniquely defines the element $M\xi(t) \in B(X)$.

In fact the Radon integral $\int_E \langle \xi_\varepsilon(t)A, g \rangle d\mu(\varepsilon)$ of bounded measurable complex-valued function on the set E by the finite additive nonnegative normalized measure μ exists and it is the bounded bilinear function on the space $X \times X_*$.

Theorem 5. If there is the dense subset $D \subset X$ in the space X such that for any $A \in D$ the family of maps $\xi_\varepsilon(t)A \in C(R_+, X)$, $\varepsilon \in E$ is weak (or strong) uniformly continuous and the family of maps ξ_ε , $\varepsilon \in E$, is uniformly bounded then for any measure $\mu \in W(E)$ the mean values of random variable ξ is continuous operator-valued function: $M\xi \in C_w(R_+, B(X))$ (or $M\xi \in C_s(R_+, B(X))$).

Proof. The weak uniform continuity means that for any $A \in D$, any $g \in X_*$ and any $\sigma > 0$ there is the constant $\delta > 0$ such that $\sup_{t \in R_+, \varepsilon \in E} |\langle \xi_\varepsilon(t + \Delta t)A - \xi_\varepsilon(t)A, g \rangle| \leq \sigma$ if $|\Delta t| < \delta$. The strong analog of this condition is following: for any $A \in D$ and any $\sigma > 0$ there is the constant $\delta > 0$ such that $\sup_{t \in R_+, \varepsilon \in E} \|\xi_\varepsilon(t + \Delta t)A - \xi_\varepsilon(t)A\|_X \leq \sigma$ if $|\Delta t| < \delta$.

The uniformly boundedness of random variable ξ means that there is the constant $C > 0$ such that $\sup_{\varepsilon \in E, t \in R_+} \|\xi_\varepsilon(t)\|_{B(X)} \leq C$.

Since the random variable ξ is uniformly bounded then for any $t \geq 0$, any $A \in X$ and any $g \in X_*$ the function $\langle \xi_\varepsilon(t)A, g \rangle$, $\varepsilon \in E$, is bounded. Therefore for any measure $\mu \in W(E)$ the integral (20) is correctly defined as the Radon integral (see [12]). Moreover the integral (20) as the function of argument g is the bounded linear functional on the space X_* . Hence for any

$A \in X$ Pettis integral $\int_{G(X)} \xi_\varepsilon A d\mu(\varepsilon) \in X$ is correctly defined. Then for arbitrary $t > 0$ the mean value $M\xi(t) = \int_{G(X)} \xi_\varepsilon d\mu(\varepsilon) \in B(X)$ is correctly defined.

According to the weak uniformly continuity condition for any $A \in D$, any $g \in X_*$ and any $\sigma > 0$ there is the constant $\delta > 0$ such that for any $t \in R_+$ and any $\Delta t \in (0, \delta) : t + \Delta t \in R_+$ the following estimates take place $\sup_{t \in R_+} |\langle (M\xi(t + \Delta t) - M\xi(t))A, g \rangle| = \sup_{t \in R_+} |\langle \int_E [\xi_\varepsilon(t + \Delta t)A - \xi_\varepsilon(t)A, g] d\mu| \leq \int_E \sup_{t \in R_+, \varepsilon \in E} |\langle \xi_\varepsilon(t + \Delta t)A - \xi_\varepsilon(t)A, g \rangle| d\mu \leq \sigma$.

If the strong uniformly continuity condition holds then for any $A \in D$ and any $\sigma > 0$ there is the constant $\delta > 0$ such that for any $t \in R_+$ and any $\Delta t \in (0, \delta) : t + \Delta t \in R_+$, the following estimates take place $\sup_{t \in R_+} \|M\xi(t + \Delta t)A - M\xi(t)A\|_X = \sup_{t \in R_+} \sup_{\|g\|_{X_*}=1} |\langle (M\xi(t + \Delta t) - M\xi(t))A, g \rangle| \leq \int_E \sup_{t \in R_+, \varepsilon \in E} \sup_{\|g\|_{X_*}=1} |\langle \xi_\varepsilon(t + \Delta t)A - \xi_\varepsilon(t)A, g \rangle| d\mu = \int_E \sup_{t \in R_+, \varepsilon \in E} \|\xi_\varepsilon(t + \Delta t)A - \xi_\varepsilon(t)A\|_X d\mu \leq \sigma$.

For any $u \in X$ there is an element $A \in D$ such that $\|u - A\|_X \leq \sigma$. Then according to uniformly boundedness condition the estimate $\|M\xi(t)u - M\xi(t)A\|_X \leq C\sigma$ holds for any $t \geq 0$. Thus the continuity of the mean value $M\xi$ of random variable ξ takes place in corresponding topologies.

Remark. The examples of families of operator-valued functions from the space $C_s(R_+, B(X))$ which satisfies the condition of dense weak (strong) uniformly continuousness are considered in the papers [18, 12].

Corollary 4. Suppose that $S \subset E$ and $s_0 \in E$ is the limit point of the set S . Let ξ be the map of the set S into the topological vector space $C_w(R_+, B(X))$ such that for any $s \in S$ the value $\xi(s) = \mathbf{U}_s \in C_w(R_+, B(X))$ is strongly continuous one-parameter semigroup.

I) If the map $\xi : S \rightarrow C_w(R_+, B(X))$ has the limit ξ_0 as $s \rightarrow s_0$ then for any measure $\mu \in W(S, s_0)$ the mean value of random variable $\xi : (S, 2^S, \mu) \rightarrow C_w(R_+, B(X))$ coincides with the limit ξ_0 .

II) If the image Ξ of the map $\xi : S \rightarrow C_w(R_+, B(X))$ is precompact then for any measure $\mu \in W_0(S, s_0)$ the mean value of random variable $\xi : (S, 2^S, \mu) \rightarrow C_w(R_+, B(X))$ is the limit of the map ξ in the point s_0 with respect to ultrafilter $F_\mu = \mu^{-1}(1)$. Conversely for any limit point $\xi_0 \in C_w(R_+, B(X))$ of the image Ξ there is the measure $\mu_0 \in W_0(S, s_0)$ such that ξ_0 is the mean value of the random variable $\xi : (S, 2^S, \mu_0) \rightarrow C_w(R_+, B(X))$.

The statement of the corollary 4 is the consequence of theorems 1 and 5. The examples of applications of corollary 4 to concrete maps is given in the papers [12].

Thus under the assumptions of the theorem 5 or corollary 4 the mean value of the random variable ξ is an element of the space $C_w(R_+, B(X))$ which will be called as the family of averaging maps of the space X .

Remark 7. The mean value of random semigroup can have no semigroup property. The

trivial example of this phenomenon is the averaging of two unitary semigroup e^{-it} , $t \geq 0$ and e^{it} , $t \geq 0$, acting in one-dimensional complex Hilbert space \mathbf{C} . If each of this semigroup has the probability $\frac{1}{2}$ then the mean value of this random semigroup is one-parameter family $F(t)$, $t \geq 0$, of maps of the space \mathbf{C} , any of each acting as the multiplier on the number $\cos t$. Since $\cos(t+s) \neq (\cos t)(\cos s)$ then the family of averaging maps $F(t)$, $t \geq 0$, is not semigroup. The paper [29] contains the example of the sequence of approximating semigroup which has the limit in the weak operator topology. The limit operator valued function for this sequence of semigroups has no semigroup property and has memory effect of dependence of the derivative on the values of the function in previous moments of time. Since limit of the sequence is the mean value of random semigroups on the measurable space with spectral measure then the mean value of random semigroups can possess the memory effects.

5 The pseudomeasures associated with the Cauchy problem

. Let symbol $\mathcal{L}(C(R_+, R^d), \mathcal{A}, \sigma_{cyl})$ note the topological vector space of pseudomeasures on the measurable space $(C(R_+, R^d), \mathcal{A})$ endowed with the topology σ_{cyl} which is generated by the cylindrical functionals. (The definition of topology σ_{cyl} is given by the equality (12)). Any map $\xi : E \rightarrow \mathcal{L}(C(R_+, R^d), \mathcal{A}, \sigma_{cyl})$ is the measurable map of the space with the measure $(E, 2^E, \mu)$ into the measurable space $(\mathcal{L}(C(R_+, R^d), \mathcal{A}, \sigma_{cyl}), \mathcal{A}_{\sigma_{cyl}})$. Any random variable $\xi : E \rightarrow \mathcal{L}(C(R_+, R^d), \mathcal{A}, \sigma_{cyl})$ is called random pseudomeasure.

Any continuous pseudomeasure ν on the algebra \mathcal{A} (which variation can be infinite) generates the two-parameter family of operators $\mathbf{V}_{t_1, t_2}^\mu$, $t_1, t_2 \in R_+$, in the Banach space $B(H)$ which is defined by the family of bounded bilinear forms β_{μ, t_1, t_2} (ñññññ. (6), (10)). Conversely any two-parameter family of bounded linear operators \mathbf{V}_{t_1, t_2} , $t_1, t_2 \in R_+$, in Hilbert space H defines the Markovian bounded pseudomeasure $\nu_{\mathbf{V}}$ on the algebra \mathcal{A} (which variation can be infinite) which is given by the equality (3).

Therefore for any random variable ξ_ε with the values in topological vector space Z the random variable ν_ε with the values in topological vector space $\mathcal{L}(C(R_+, R^d), \mathcal{A}, \sigma_{cyl})$ corresponds in accordance with (3).

Moreover the convergence of the sequence of operators ξ_ε , $\varepsilon \in E$, as $\varepsilon \rightarrow \varepsilon_0$ in the strong (weak) operator topology is connected with the convergence of the sequence of pseudomeasures in the space $\mathcal{L}(C(R_+, R^d), \mathcal{A}, \sigma_{cyl})$ by the theorems 2, 3.

5.1 The averaging procedure for the sequence of pseudomeasures.

The sequence of the values of the measures μ_n on some elements of the family of the cylindrical subspace $A = A_{B_1, \dots, B_m}^{t_1, \dots, t_m}$, $m \in \mathbf{N}$ can diverges. To investigate the limit behavior of the sequence of pseudomeasures μ_n and to obtain the limit function of a set on the algebra \mathcal{A} the methods of the Banach limits and invariant means (see [1], [3] [25], [15], [16], [18]) have being used. For this aim the measurable space of regularizing parameters $(\mathbf{N}, 2^{\mathbf{N}})$ endowed with the nonnegative normalized finitely additive measure $\varrho \in ba(\mathbf{N}, 2^{\mathbf{N}}) = l_{\infty}^*$, which is concentrated in the arbitrary neighborhood of the point $\infty = \sup(\mathbf{N})$. The existence of a nontrivial class of measures with the above property and its studying had being published in the papers [9, 15].

The procedure of the averaging of the sequence of pseudomeasures μ_n with values in the space $\mathcal{L}(C(R_+, R), \mathcal{A})$ by measure ϱ is given by the Pettis integral. The averaging function of a set μ^{ϱ} on the algebra \mathcal{A} is given by the relation $\mu^{\varrho} = \int_{\mathbf{N}} \mu_n d\varrho(n)$:

$$\mu^{\varrho}(A_{B_1, \dots, B_m}^{t_1, \dots, t_m}) = \int_{\mathbf{N}} \mu_n(A_{B_1, \dots, B_m}^{t_1, \dots, t_m}) d\mu(n) \quad \forall A_{B_1, \dots, B_m}^{t_1, \dots, t_m} \in \mathcal{A}.$$

The averaging function of a set μ^{ϱ} is defined on the algebra \mathcal{A} and it is additive function, i.e. μ^{ϱ} is pseudomeasure.

Definition 5. The random variable with the values in the measurable space $(\mathcal{L}(C(R_+, R^d), \mathcal{A}), \sigma_{cyl})$ of pseudomeasures is the measurable map of the measurable space $(E, 2^E)$ endowed with the measure $\nu \in W(E)$ into the topological vector space $(\mathcal{L}(C(R_+, R^d), \mathcal{A}), \sigma_{cyl})$.

Definition 6. The random pseudomeasure $m : (E, 2^E, \nu) \rightarrow \mathcal{L}(C(R_+, R^d), \mathcal{A}, \sigma_{cyl})$ is called weakly iniformly bounded if for any $A \in \mathcal{A}$ there is the constant $M > 0$ and the set $E(A) \subset E$ such that $|m_{\varepsilon}(A)| \leq M$ for all $\varepsilon \in E(A)$ and $\nu(E(A)) = 1$.

Suppose that $\nu \in W(E)$ and for any $\varepsilon \in E$ the pseudomeasure μ_{ε} is defined on the algebra \mathcal{A} of subsets of the space $\Omega = C(R_+, R^d)$. Let the function of a set $\nu * \mu$ be defined by the equality $\nu * \mu(A_E \times A_{\Omega}) = \int_{A_E} \mu_{\varepsilon}(A_{\Omega}) d\nu(\varepsilon)$ for arbitrary $A_E \in 2^E$ and $A_{\Omega} \in \mathcal{A}$.

Lemma 3. If $\nu \in W(E)$ and random pseudomeasure μ on algebra \mathcal{A} is weakly uniformly bounded, then the function of a set $\nu * \mu$ is defined on the algebra $2^E \otimes \mathcal{A}$ and is finite additive.

In fact for any $A_{\Omega} \in \mathcal{A}$ the numerical function $\mu_{\varepsilon}(A_{\Omega})$, $\varepsilon \in E$, is defined on the set E and is bounded on the set $E(A_{\Omega})$. Since $\nu \in W(E)$ and $\nu(E(A_{\Omega})) = 1$ then the function $\mu_{\varepsilon}(A_{\Omega})$, $\varepsilon \in E$, is integrable with respect to measure ν . Hence the function $\nu * \mu$ is defined on the following family sets $\{A_E \times A_{\Omega}; A_E \in 2^E, A_{\Omega} \in \mathcal{A}\}$ and this function satisfies the properties of additivity with respect to both arguments. In fact, the additivity property of the function $\nu * \mu$ with respect to the first argument is the consequence of the additivity of Pettis the minimal algebra containing the minimal algebra containing the minimal algebra containing

integral. It's additivity with respect to the second argument is the consequence of linearity of Pettis integral and additivity of pseudomeasures μ_ε . Hence the function $\nu * \mu$ has the unique continuation by the standard scheme onto the finite additive function of a set on the minimal algebra containing the family sets $\{A_E \times A_\Omega; A_E \in 2^E, A_\Omega \in \mathcal{A}\}$, i.e. on the algebra $2^E \times \mathcal{A}$.

Definition 7. The mean value of random pseudomeasure μ on the space with the measure $(E, 2^E, \nu)$ is defined as the pseudomeasure μ^ν on the algebra \mathcal{A} such that $\mu^\nu(A) = \int_E \mu_\varepsilon(A) d\nu(\varepsilon) \forall A \in \mathcal{A}$.

Since the pseudomeasure μ^ν is equal to the restriction of pseudomeasure $\nu * \mu$ onto the subalgebra $E \otimes \mathcal{A}$ then the pseudomeasure $\nu * \mu$ is Young pseudomeasure with respect to pseudomeasure μ^ν in the sense of definition 3.5.7 of [8].

Lemma 4. Suppose that $\nu \in W(E)$. If the random pseudomeasure μ on the space $(E, 2^E, \nu)$ is weakly uniformly bounded then its mean value is the element of the space $\mathcal{L}(C(R_+, R^d), \mathcal{A})$.

In fact the pseudomeasure μ^ν is defined on the algebra \mathcal{A} and is additive function on this algebra. It should be noted that the pseudomeasure μ^ν is equal to the restriction of pseudomeasure $\nu * \mu$ on the subalgebra $E \otimes \mathcal{A}$.

Remark 8. If for any $\varepsilon \in E$ the pseudomeasure μ_ε is continuous (see definition 1), then the family of maps $\mathbf{V}_{\mu_\varepsilon}$ is the family of bounded linear operators in the space X . Moreover if for any $\varepsilon \in E$ the pseudomeasure μ_ε is Markovian and stationary then the family of maps $\mathbf{V}_{\mu_\varepsilon}$ is oneparameter semigroup of bounded linear operators in the space X . However even the measure μ_ε is continuous, stationary and Markovian for any $\varepsilon \in E$ but the averaging measure μ^ν can have no Markovian property and has no continuity property.

6 Topologies on the space of pseudomeasures generated by the nonlinear cylindric functional.

The topology in the space of Markovian continuous pseudomeasures (see [17]) which is corresponded by the convergence of semigroup in the space H is generated by the family of the cylindrical functional $P_A(\mu)$, $\mu \in \mathcal{L}(C(R_+, R^d), \mathcal{A})$ which is parametrized by the family of sets $\{A \in \mathcal{A}\}$. In particular if the functional P_A is given by some $A \in Cyl_2$ then the linear functionals P_A is given by (13).

Let us introduce the nonlinear functionals on the space of pseudomeasures which is defined by using of quantum states. The symbol $\Sigma(H)$ notes the set of quantum states (see [1] [19], [14]) which is the intersection of the unit sphere with the positive cone of Banach space $(B(H))^*$ (here $(B(H))^*$ is conjugate to the space $B(H)$ of bounded linear operators).

If μ is continuous pseudomeasure $\mu \in \mathcal{L}_{cont}(C(R_+, R^d), \mathcal{A})$ then for any nonnegative numbers $t_1, t_2 \in R_+$ the bounded sesquilinear form $\beta_\mu^{t_1, t_2}$ defines two-parametric family of bounded linear

operators $\mathbf{U}_\mu^{t_0, t_1}$, $t_0, t_1 \in R_+$, which is given by the pseudomeasure μ in accordance with the equalities (6). For any bounded linear self-adjoint operator \mathbf{A} with the discrete spectrum $\sigma(\mathbf{A}) = \{a_k\}$ which has the orthonormal basis $\{\psi_k\}$ of eigen vectors, for any $u \in H$ and any $t_1, t_2 \in R_+$ the following number is defined

$$F_{\rho_u, \mathbf{A}}^{t_0, t_1}(\mu) = \sum_{k=1}^{\infty} a_k \left| \int_{C(R_+, R)} u(\xi(t_0)) \bar{\psi}_k(\xi(t_1)) d\mu(\xi) \right|^2 = \sum_{k=1}^{\infty} a_k |(u, \mathbf{U}_\mu^{t_0, t_1} \psi_k)|^2$$

for arbitrary continuous measure $\mu \in \mathcal{L}_{cont}(C(R_+, R^d), \mathcal{A})$.

Since the set of bounded linear operators with the discrete spectrum is dense in the space of bounded linear operators then for any $t_0, t_1 \geq 0$, $\mu \in \mathcal{L}_{cont}(C(R_+, R^d), \mathcal{A})$ and $u \in H$, $\|u\| = 1$ the functional $F_{\rho_u, \cdot}^{t_0, t_1}(\mu) = \langle \rho_u, \mathbf{U}_\mu^{t_0, t_1} (\mathbf{U}_\mu^{t_0, t_1})^* \rangle$ can be defined according to continuity on the whole space $B(H)$. Since any element $\rho \in \Sigma(H)$ can be decomposed into the Pettis integral on the set of pure vector states (see [1, 14]) then for any fixed values $t \geq 0$, $\mathbf{A} \in B(H)$ and $\mu \in \mathcal{L}_c(C(R_+, R^d), \mathcal{A})$ the functional $F_{\rho, \mathbf{A}}^{t_0, t_1}(\mu)$ can be continued according to linearity on the whole set $\Sigma(H)$ by the rule: if the state $\rho \in \Sigma(H)$ can be decomposed by Pettis integral $\rho = \int_{\|e\|=1} \rho_e d\lambda(e)$ with respect to some nonnegative normalised finite additive measure λ on the unite sphere $(S_1(H), 2^{S_1(H)})$ of Hilbert space H then the value of functional $F_{\rho, \mathbf{A}}^{t_0, t_1}$ on the pseudomeasure $\mu \in \mathcal{L}_{cont}(C(R_+, R^d), \mathcal{A})$ is equal $F_{\rho, \mathbf{A}}^{t_0, t_1}(\mu) = \int_{\|e\|=1} F_{\rho_e, \mathbf{A}}^{t_0, t_1}(\mu) d\lambda(e)$.

Thus if $\mu \in \mathcal{L}_{cont}(C(R_+, R^d), \mathcal{A})$ is continuous pseudomeasure generating the two-parameter family $\mathbf{U}_\mu^{t_1, t_2}$, $t_1, t_2 \in R_+$, of bounded linear operators then for any $\mathbf{A} \in B(H)$, $\rho \in \Sigma(H)$; $t_1, t_2 \in R_+$ the following equality holds

$$F_{\rho, \mathbf{A}}^{t_1, t_2}(\mu) = (\rho, \mathbf{U}_\mu(t_2 - t_1) \mathbf{A} \mathbf{U}_\mu^{-1}(t_2 - t_1)), \quad \forall t_1, t_2 : t_2 \geq t_1 \geq 0, \rho \in \Sigma(H), \mathbf{A} \in B(H). \quad (21)$$

For any $\rho \in \Sigma(H)$, $t_1, t_2 \in R_+$ and $\mathbf{A} \in B(H)$ the functional (21) $F_{\rho, \mathbf{A}}^{t_0, t_1}$ is the quadratic functional on the space $\mathcal{L}(C(R_+, R^d), \mathcal{A})$, which is defined on some subset of the space $\mathcal{L}(C(R_+, R^d), \mathcal{A})$.

The symbol \mathcal{F} notes the family of functionals $\{F_{\rho, \mathbf{A}}^{t_1, t_2}, t_1, t_2 \in R_+, \rho \in \Sigma(H), \mathbf{A} \in B(H)\}$. Any functional of this family is defined on some subset of linear space $\mathcal{L}(C(R_+, R^d), \mathcal{A})$.

The symbol \mathcal{D} notes the set $\mathcal{D} \subset \mathcal{L}(C(R_+, R^d), \mathcal{A})$ of pseudomeasures of the space $\mathcal{L}(C(R_+, R^d), \mathcal{A})$ which belongs to the definition domain of all functionals of the family \mathcal{F} . As it is shown above the family of the functionals $\{F_{\rho, \mathbf{A}}^{t_1, t_2}; t_1, t_2 \in R_+, \rho \in \Sigma(H), \mathbf{A} \in B(H)\}$ is defined on the space $\mathcal{L}_{cont}(C(R_+, R^d), \mathcal{A})$, i.e. $\mathcal{L}_{cont}(C(R_+, R^d), \mathcal{A}) \subset \mathcal{D}$. On the other side if $\mu \in \mathcal{L}(C(R_+, R^d), \mathcal{A})$ is some pseudomeasure such that any functional (21) is defined on this pseudomeasure then the bilinear form $\beta_\mu^{t_0, t_1}$ must be bounded on the space H_S for arbitrary $t_0, t_1 \geq 0$. Therefore the set \mathcal{D} is the set of pseudomesures satisfying the conditions (5a) for $m = 1$ and any $t_0, t_1 \geq 0$.

Lemma 5. If $\mu, \nu \in \mathcal{D}$ then $\mu + \nu \in \mathcal{D}$ and $\alpha\mu \in \mathcal{D}$ for any $\alpha \in \mathbf{C}$.

Since $\int_{C(R_+, R)} u(\xi(t_0))\bar{\psi}_k(\xi(t_1))d(\mu + \nu)(\xi) = \int_{C(R_+, R)} u(\xi(t_0))\bar{\psi}_k(\xi(t_1))d(\mu)(\xi) + \int_{C(R_+, R)} u(\xi(t_0))\bar{\psi}_k(\xi(t_1))d(\nu)(\xi)$ then the following inequality holds $|\int_{C(R_+, R)} u(\xi(t_0))\bar{\psi}_k(\xi(t_1))d(\mu + \nu)(\xi)|^2 \leq 2(|\int_{C(R_+, R)} u(\xi(t_0))\bar{\psi}_k(\xi(t_1))d(\mu)(\xi)|^2 + |\int_{C(R_+, R)} u(\xi(t_0))\bar{\psi}_k(\xi(t_1))d(\nu)(\xi)|^2)$. Thus for any values $t_1, t_2, \mathbf{A}, \rho$ and any pseudomeasures $\mu, \nu \in \mathcal{D}$ the following inequality holds:

$$F_{\rho, \mathbf{A}}^{t_1, t_2}(\mu + \nu) \leq 2(F_{\rho, \mathbf{A}}^{t_1, t_2}(\mu) + F_{\rho, \mathbf{A}}^{t_1, t_2}(\nu)). \quad (22)$$

Therefore the values of the functionals $F_{\rho, \mathbf{A}}^{t_1, t_2}(\mu + \nu)$ are defined for any $t_1, t_2 \in R_+$, $\rho \in \Sigma(H)$, $\mathbf{A} \in B(H)$ because the values of functionals $F_{\rho, \mathbf{A}}^{t_1, t_2}(\mu)$ and $F_{\rho, \mathbf{A}}^{t_1, t_2}(\nu)$ are defined for any $t_1, t_2 \in R_+$, $\rho \in \Sigma(H)$, $\mathbf{A} \in B(H)$.

The second statement of the lemma is the consequence of the equality $F_{\rho, \mathbf{A}}^{t_1, t_2}(\alpha\mu) = \|\alpha\|^2 F_{\rho, \mathbf{A}}^{t_1, t_2}(\mu)$ which holds according to (22) for any values of $t_1, t_2, \mathbf{A}, \rho$.

Corollary 5. The set \mathcal{D} is the linear subspace in the space $\mathcal{L}(C(R_+, R^d), \mathcal{A})$ such that $\mathcal{L}_{cont}(C(R_+, R^d), \mathcal{A}) \subset \mathcal{D} \subset \mathcal{L}(C(R_+, R^d), \mathcal{A})$.

Let the topology $\tau_{\mathcal{F}}$ on the linear space \mathcal{D} be defined by the family of functionals \mathcal{F} in the following sence: the topology $\tau_{\mathcal{F}}$ is the least toplogy containing the sets $\{f^{-1}(B), f \in \mathcal{F}, B \in \mathcal{B}(\mathbf{C})\}$ where $\mathcal{B}(\mathbf{C})$ is Borel algebra (but not only system of open sets) of subsets of comlex plane \mathbf{C} . If the set $A_{f, B} \in \tau_{\mathcal{F}}$ for some $f \in \mathcal{F}$, $B \in \mathcal{B}(\mathbf{C})$ is defined by the condition $A_{f, B} = \{\mu \in \mathcal{D} : f(\mu) \in B\}$ then the set $\mathcal{D} \setminus A_{f, B} = \{\mu \in \mathcal{D} : f(\mu) \notin B\} = A_{f, B^c}$ belongs to the topology $\tau_{\mathcal{F}}$. Thus the topology $\tau_{\mathcal{F}}$ is closed with respect to complement operation (any open set of the topology $\tau_{\mathcal{F}}$ is closed set).

Let symbol $\Sigma_{\mathcal{F}}$ note the algebra of subsets of the space \mathcal{D} which is generated by the family of functionals \mathcal{F} i.e. $\Sigma_{\mathcal{F}}$ is the least algebra containing the sets $\{f^{-1}(B), f \in \mathcal{F}, B \in \mathcal{B}(\mathbf{C})\}$. Then the following condition $\Sigma_{\mathcal{F}} \subset \tau_{\mathcal{F}}$ holds because the topology $\tau_{\mathcal{F}}$ is closed with respect to complement operation.

Remark 9. The topolotgical subspace $(\mathcal{L}_{cm}(C(R_+, R^d), \mathcal{A}), \tau_{\mathcal{F}})$ of continuous Markovian pseudomeasures in the topological space $(\mathcal{D}, \tau_{\mathcal{F}})$ is Hausdorff space. In fact, due to Markovian property (5.6) there is the one-to-one correspondence between the set $\mathcal{L}_{cm}(C(R_+, R^d), \mathcal{A})$ and the set of two-parametric families of bounded linear operators subjecting to dynamical property $\mathbf{U}^{t_2, t_3} \mathbf{U}^{t_1, t_2} = \mathbf{U}^{t_1, t_3}$, $0 \leq t_1 \leq t_2 \leq t_3$. And if $(\rho, \mathbf{U}_{\mu_1}^{t_1, t_2} \mathbf{A}(\mathbf{U}_{\mu_1}^{t_1, t_2})^*) = (\rho, \mathbf{U}_{\mu_2}^{t_1, t_2} \mathbf{A}(\mathbf{U}_{\mu_2}^{t_1, t_2})^*)$ for all $t_1, t_2 : t_2 \geq t_1 \geq 0$, $\rho \in \Sigma(H)$, $\mathbf{A} \in B(H)$, then $\mu_1 = \mu_2$ because any Markovian pseudomeasure can be uniquely reconstructed by its restriction on the class Cyl_2 of cylindrical subsets with two-dimentional base. But the hole space $(\mathcal{D}, \tau_{\mathcal{F}})$ is not Hausdorff space since there is some two continuous (but not Markovian) pseudomeasures μ_1, μ_2 such that it have the common

restrictions on the class Cyl_2 and distinguishing restrictions on the class Cyl_3 .

For any pseudomeasure $\mu_0 \in \mathcal{D}$ the following class of pseudomeasures $K_{\mu_0} = \{\mu \in \mathcal{D} : f(\mu) = f(\mu_0) \forall f \in \mathcal{F}\}$ is defined. The class of pseudomeasures $K_\theta = \{\mu \in \mathcal{D} : f(\mu) = 0 \forall f \in \mathcal{F}\}$ is the linear subspace in the linear \mathcal{D} according to the lemma 5 (see 22)). Hence the factor-space \mathcal{D}/K_θ is Hausdorff topological vector subspace in the space \mathcal{D} . Since any functional of the family \mathcal{F} has the constant values on the sets $K_{\mu_0}, \mu_0 \in \mathcal{D}$ then any own (proper) subset of a set $K_{\mu_0}, \mu_0 \in \mathcal{D}$ is no element of the topology $\tau_{\mathcal{F}}$. Hence any measurable numerical function on the measurable space $(\mathcal{D}, \Sigma_{\mathcal{F}})$ has the constant value on the sets $K_{\mu_0}, \mu_0 \in \mathcal{D}$. I.e. if $\mathcal{D}_\theta = \mathcal{D} \setminus K_\theta$ then any measurable function on the space $(\mathcal{D}, \Sigma_{\mathcal{F}})$ takes constant values on the sets $K_{\mu_0}, \mu_0 \in \mathcal{D}$. Therefore any measurable function on the space $(\mathcal{D}, \Sigma_{\mathcal{F}})$ is the function on the factor-space $(\mathcal{D}_\theta, \Sigma_{\mathcal{F}})$.

Let symbol $(B(\mathcal{D}, \Sigma_{\mathcal{F}}), \tau_D)$ note the topological vector space of measurable functions on the topological vector space $(\mathcal{D}, \tau_{\mathcal{F}})$. The topology τ_D in the space $(B(\mathcal{D}, \Sigma_{\mathcal{F}}), \tau_D)$ is generated by the family of functionals $\varphi_\mu, \mu \in \mathcal{D}$, any of each acting by the formular $\varphi_\mu(f) = f(\mu), f \in B(\mathcal{D}, \Sigma_{\mathcal{F}})$. It should be noted that the functions of family \mathcal{F} (but not only it) belong to the topological vector space $(B(\mathcal{D}, \Sigma_{\mathcal{F}}), \tau_D)$.

Lemma 6. *If the topology τ_B on the linear space \mathcal{D} is generated by the family of functionals $B(\mathcal{D}, \Sigma_{\mathcal{F}})$ then the topology τ_B (is equivalent to) coincides with the topology $\tau_{\mathcal{F}}$.*

It is obvious that $\tau_{\mathcal{F}} \subset \tau_B$. If we assume that the topology τ_B is more wide than the topology $\tau_{\mathcal{F}}$, then (according to definition of the space $B(\mathcal{D}, \Sigma_{\mathcal{F}})$) there is the measurable (with respect to the algebra $\Sigma_{\mathcal{F}}$) functional $g \in B(\mathcal{D}, \Sigma_{\mathcal{F}})$ and there is the Borel subset $B \in \mathcal{B}(\mathbb{C})$ such that the set $g^{-1}(B)$ is not belong to the topology $\tau_{\mathcal{F}}$. But it is the contradiction to the measurability of the functional g with respect to the algebra $\Sigma_{\mathcal{F}}$ because $g^{-1}(B) \in \Sigma_{\mathcal{F}}$ for any $B \in \mathcal{B}(\mathbb{C})$ and $\Sigma_{\mathcal{F}} \subset \tau_{\mathcal{F}}$. The proposition is proved.

Let us introduce the functional \mathcal{V} on the linear space \mathcal{D} which values is given by the formular $\mathcal{V}(\mu) = \sup_{\|\mathbf{A}\|=1, t_1, t_2 \in \mathbb{R}^+, \rho \in \Sigma(H)} |F_{\rho, \mathbf{A}}^{t_1, t_2}(\mu)|$ for any $\mu \in \mathcal{D}$. The functional \mathcal{V} is defined on the linear space \mathcal{D} and it is the seminorm on the space \mathcal{D} according to the inequality (22). Then the functional \mathcal{V} is the norm on the factor-space $\mathcal{D}_\theta = \mathcal{D} \setminus K_\theta$.

Let symbol $b(\mathcal{D}, \Sigma_{\mathcal{F}})$ note the Banach space of bounded measurable functions on the space \mathcal{D} endowing with the norm $\|F\| = \sup_{\mu \in \mathcal{D}: \mathcal{V}(\mu) \leq 1} |F(\mu)|$. Then $\mathcal{F} \subset b(\mathcal{D}, \Sigma_{\mathcal{F}})$ and the following statement holds.

Lemma 7. *If the topology τ_b on the linear space \mathcal{D} is generated by the family of functionals $b(\mathcal{D}, \Sigma_{\mathcal{F}})$ then the topology τ_b coincides with the topology τ_B .*

The topology τ_b is no wider than the topology τ_B because $b(\mathcal{D}, \Sigma_{\mathcal{F}}) \subset B(\mathcal{D}, \Sigma_{\mathcal{F}})$. On the other side for any function $f \in B(\mathcal{D}, \Sigma_{\mathcal{F}})$ and any interval $\Delta \in \mathbb{R}$ there is the function $\varphi \in b(\mathcal{D}, \Sigma_{\mathcal{F}})$ (which can have only two different values on the set $f^{-1}(\Delta)$ and its complement)

and the interval $\Delta' \in R$ such that $f^{-1}(\Delta) = \varphi^{-1}(\Delta')$. Hence the topology τ_B is no wider (stronger) than the topology τ_b .

Remark 10. If $\Phi \in B(\mathcal{D}, \Sigma_{\mathcal{F}})$ (i.e. $\Phi^{-1}(B) \in \Sigma_{\mathcal{F}}$ for any $B \in \mathcal{B}(\mathbf{C})$), then $\Phi(\mu) = \Phi(\mu_0) \forall \mu \in K_{\mu_0}$.

Let symbol $ba(\mathcal{D}, \Sigma_{\mathcal{F}})$ note the Banach space of measures with finite variation on the measurable space $(\mathcal{D}, \Sigma_{\mathcal{F}})$, i.e. the conjugate space for Banach space $b(\mathcal{D}, \Sigma_{\mathcal{F}})$ (see [5]). The measure $m \in ba(\mathcal{D}, \Sigma_{\mathcal{F}})$ is called as the limit point of the sequence of the measures $\{m_n\}$ in the space $ba(\mathcal{D}, \Sigma_{\mathcal{F}})$ if for any $\epsilon > 0$ and any $F \in b(\mathcal{D}, \Sigma_{\mathcal{F}})$ the inclusion $F(m_n) \in O_{\epsilon}(F(m))$ holds for any m from some infinite subset $\mathbf{N}_{\epsilon, F}$ of the set \mathbf{N} .

Let $E = (0, 1)$ be the set of regularization parameters and ξ is the map $\xi : E \rightarrow \mathcal{D}$ such that the value $\xi(\varepsilon)$ on any point $\varepsilon \in E$ is the pseudomeasure μ_{ε} on the algebra \mathcal{A} which is generated by the unitary semigroup \mathbf{U}_{ε} in accordance with the equality (5). Then for any $\varepsilon \in E$ the value μ_{ε} of the map ξ satisfies the condition $\mu_{\varepsilon} \in \mathcal{L}_{cm}(C(R_+, R^d), \mathcal{A})$ and hence $\mu_{\varepsilon} \in \mathcal{D}$.

Let the measure $\delta(\mu - \mu_0) \in ba(\mathcal{D}, \Sigma_{\mathcal{F}})$ be defined by the condition: for any pseudomeasure $\mu_0 \in \mathcal{D}$ the equality $\langle \delta(\mu - \mu_0), f \rangle = f(\mu_0)$ holds for any $f \in b(\mathcal{D}, \Sigma_{\mathcal{F}})$. Since the function $f \in b(\mathcal{D}, \Sigma_{\mathcal{F}})$ is measurable then it has the constant values on the sets K_{μ_0} , $\mu_0 \in \mathcal{D}$. Therefore $\delta(\mu - \mu_0) = \delta(\mu - \mu_1)$ if $\mu_1 \in K_{\mu_0}$.

Theorem 6. The set $M \subset ba(\mathcal{D}, \Sigma_{\mathcal{F}})$ of values of the sequence of measures $\delta(\mu - \mu_{\varepsilon})$, $\varepsilon \in E$, is compact in the topology τ_b .

If $\hat{\nu} \in ba(\mathcal{D}, \Sigma_{\mathcal{F}})$ is the limit point of the sequence of measures $\delta(\mu - \mu_{\varepsilon})$, $\varepsilon \in E$, in the space $ba(\mathcal{D}, \tau_b)$ then the equality $F(\mu_{\varepsilon}) \rightarrow \langle F(\mu) \rangle_{\nu} = \int_{\mathcal{D}} F(\mu) d\nu(\mu)$ holds for any functional $F \in b(\mathcal{D}, \Sigma_{\mathcal{F}})$.

The statement of this theorem is the realization of the theorem 1 statement for the spaces $Z = (b(\mathcal{D}, \Sigma_{\mathcal{F}}))^* = ba(\mathcal{D}, \Sigma_{\mathcal{F}})$ and $S = b(\mathcal{D}, \Sigma_{\mathcal{F}})$ and for the map $G : E \rightarrow Z$ acting by the equality $G(\varepsilon) = \delta(\mu - \mu_{\varepsilon})$, $\varepsilon \in E$.

In fact for any two-valued measure $\nu \in W_0(E)$ the functional $g(F) = \int_E F(\mu_{\varepsilon}) d\nu(\varepsilon)$ is linear and continuous functional on the space $b(\mathcal{D}, \Sigma_{\mathcal{F}})$. Then the sequence of the measures $\{\delta(\mu - \mu_{\varepsilon}), \varepsilon \in E\}$ converges by the ultrafilter $F_{\nu} = \nu^{-1}(1)$ to the measure ν in the following sense: for any functional $F \in b(\mathcal{D}, \Sigma_{\mathcal{F}})$ the equality $\lim_{\varepsilon \rightarrow 0, F_{\nu}} F(\mu_{\varepsilon}) = \langle F(\mu) \rangle_{\nu} = \int_E F(\mu_{\varepsilon}) d\nu(\varepsilon)$ holds.

Conversely if for some ultrafilter F of the set E with the limit point 0 the equality $\lim_{\varepsilon \rightarrow 0, \varepsilon \in F} F(\mu_{\varepsilon}) = g(F)$ holds for any functional $F \in b(\mathcal{D}, \Sigma_{\mathcal{F}})$ then there is the measure $\nu \in ba(\mathcal{D}, \Sigma_{\mathcal{F}})$ on the space \mathcal{D} such that for any functional $F \in b(\mathcal{D}, \Sigma_{\mathcal{F}})$ the equality $g(F) \rightarrow \langle F(\mu) \rangle_{\nu} = \int_{\mathcal{D}} F(\mu) d\nu(\mu)$ holds (the measure ν can be the image under the action of the map $\xi : E \rightarrow \mathcal{D}$; $\xi(\varepsilon) = \mu_{\varepsilon}$ of the measure on the set E which is generated by the

ultrafilter F). Theorem 6 is proved.

Thus the description of the limit behavior of the class of continuous (not only linear) functionals on the space of pseudomeasures \mathcal{D} is given by the measure ν on the space \mathcal{D} . In particular the description of the limit behavior of continuous linear functionals of the space of pseudomeasures \mathcal{D} (which is generated by the sequence of unitary groups in Hilbert space of quantum system H) is given by the pseudomeasure $\mu_\nu \in \mathcal{D}$ which is barycenter of measure ν .

The measure ν on the measurable space of pseudomeasures $(\mathcal{L}_{cm}(C(R_+, \Omega), \tau_{\mathcal{F}}))$ is the analog of the family of Young measures (see [11, 26]). In fact let $\Omega \in R^d$ be some bounded domain with the smooth boundary. Let the strongly continuous semigroup of unitary operators in Hilbert space $H = L_2(\Omega)$ be defined for any value of parameter $\varepsilon \in E = (0, 1)$. Suppose that $\nu \in W_0(E)$. According to the equality (5) any semigroup \mathbf{U}_ε of maps of the space H defines the unique pseudomeasure $\mu_\varepsilon \in \mathcal{L}_{cm}(C(R_+, \Omega))$. Suppose that there is the vector $u \in H$ such that for any $\varepsilon \in E$ and any $t \geq 0$ the function $u_\varepsilon(t) = \mathbf{U}_{\mu_\varepsilon}(t)u$ is uniformly continuous on the domain Ω (the examples of such situations are described in the papers [12, 15]).

Let us consider the class \mathcal{F}_x of functionals $\{\Phi_{t,x,\phi}, (t, x) \in [0, T] \times \Omega, \phi \in C(\mathbf{C})\}$ on the space \mathcal{D} which are given by the equalities $\Phi_{t,x,\phi,u} = \phi(u_\mu(t, x))$ where $u_\mu(t, \cdot) = (\mathbf{U}_\mu(t)u)(\cdot)$. Then the measure ν on the sequence of pseudomeasures $\{\mu_\varepsilon\}$, $\varepsilon \in E$, $\varepsilon \rightarrow 0$, uniquely defines the Young measure i.e. the family of measures $\{\nu_{t,x}, (t, x) \in [0, T] \times \Omega\}$ on the complex plane \mathbf{C} such that for any $f \in C_0(\mathbf{C})$ and any $(t, x) \in [0, T] \times \Omega$ the equality $\lim_{\varepsilon \rightarrow 0, \varepsilon \in F_\mu} \int_E f(u_\varepsilon(t, x)) d\mu(\varepsilon) = \int_{\mathbf{C}} f(\xi) d\nu_{t,x}(\xi)$ holds (see [26, 15]). Thus the measure on the space of pseudomeasures $\mathcal{L}_c(C(R_+, R^d), \mathcal{A})$ induces the Young measure corresponding to the sequence of approximations of solution of initial-boundary value problem.

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